§ 4.8 Nonhomogeneous Linear Systems

$$x_{n+1} = ax_n + by_n + h$$
$$y_{n+1} = cx_n + dy_n + k$$

$$\Rightarrow \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

$$\vec{x}_{n+1} = A \vec{x}_n + \vec{b}$$

Assume the system has a unique fixed point $\vec{p} = \begin{bmatrix} x \\ y \end{bmatrix}$

That is,

$$\vec{p} = A\vec{p} + b$$
$$\vec{0} = A\vec{p} + b - \vec{p}$$

Change of variable: $\vec{x}_n = \vec{y}_n + \vec{p}$ $\left(\vec{x}_{n+1} = \vec{y}_{n+1} + p \right)$

$$\sim \quad \vec{y}_{n+1} + \vec{p} = A(\vec{y}_n + \vec{p}) + b$$
$$= A\vec{y}_n + A\vec{p} + b$$

$$\vec{y}_{n+1} = A\vec{y}_n + \underbrace{A\vec{p} + b - \vec{p}}_{\vec{0}}$$

$$\vec{y}_{n+1} = A\vec{y}_n$$

$\therefore$ $\vec{y}_n$ evolves according to a homogeneous linear system.

We understand the dynamics of $\vec{y}_n$ depend only on eigenvalues of A.

$\vec{x}_n = \vec{y}_n + \vec{p}$

$\llcorner$ dynamics of $\vec{x}_n$ are the same,

only translated by $\vec{p}$.

IN SUMMARY:

SUPPOSE A HAS EIGENVALUES r & s.

THE ORIGIN IS FIXED POINT OF

$$y_{n+1} = A y_n.$$

- THE ORIGIN IS A SINK IF

$|r| < 1, \; |s| < 1$

- THE ORIGIN IS A SOURCE IF

$|r| > 1, \; |s| > 1$

- THE ORIGIN IS A SADDLE POINT IF

$|r| < 1, \; |s| > 1$.

$\vec{p}$ IS FIXED POINT OF

$$\vec{x}_{n+1} = A\vec{x}_n + \vec{b}$$

- $\vec{p}$ IS A SINK IF

$|r| < 1, \; |s| < 1$

- $\vec{p}$ IS A SOURCE IF

$|r| > 1, \; |s| > 1$

- $\vec{p}$ IS A SADDLE POINT IF

$|r| < 1, \; |s| > 1$.

In Exercises 1–10 find the fixed point of the given system and determine whether it is a sink, source or saddle.

1. $P_{n+1} = P_n + 0.4D_n - 10$
   $D_{n+1} = -0.5P_n + D_n + 2$

2. $P_{n+1} = \frac{2}{3}P_n - \frac{1}{3}Q_n + 4000$
   $Q_{n+1} = \frac{1}{6}P_n + \frac{1}{6}Q_n + 5000$

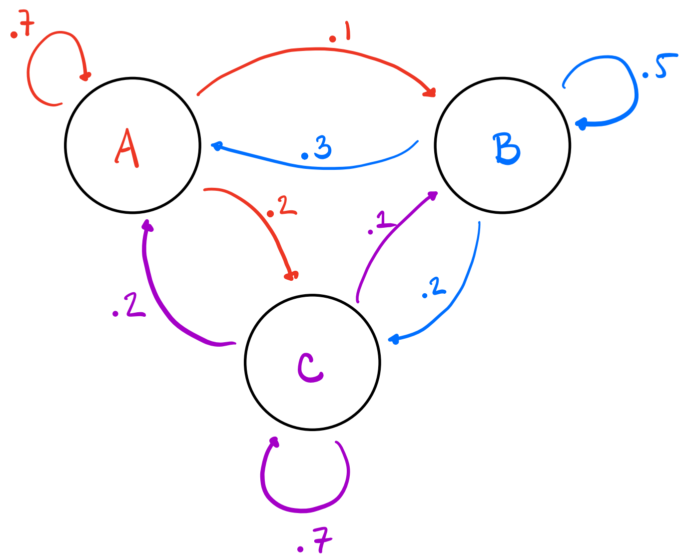
6. $x_{n+1} = 3x_n + 2y_n + 2$
   $y_{n+1} = -x_n + y_n - 2$

7. $P_{n+1} = P_n + 0.25D_n + 3$
   $D_{n+1} = -0.5P_n + D_n + 5$

Three State Markov Systems

Suppose a city with fixed population of 100,000 has 3 boroughs : A, B, C.

Each year a proportion from each borough move to another borough according to the transition diagram.



$$\Rightarrow \quad \begin{aligned} A_{n+1} &= .7A_n + .3B_n + .2C_n \\ B_{n+1} &= .1A_n + .5B_n + .1C_n \\ C_{n+1} &= .2A_n + .2B_n + .7C_n \end{aligned}$$

Since $A_n + B_n + C_n = 100{,}000 \Rightarrow C_n = 100{,}000 - A_n - B_n$

$$\Rightarrow \quad \begin{aligned} A_{n+1} &= .7A_n + .3B_n + .2(100{,}000 - A_n - B_n) \\ B_{n+1} &= .1A_n + .5B_n + .1(100{,}000 - A_n - B_n) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} A_{n+1} &= (.7 - .2)A_n + (.3 - .2)B_n + 20{,}000 \\ B_{n+1} &= (.1 - .1)A_n + (.5 - .1)B_n + 10{,}000 \end{aligned}$$

$$\Rightarrow \quad A_{n+1} = .5\,A_n + .1\,B_n + 20{,}000$$
$$B_{n+1} = 0\,A_n + .4\,B_n + 10{,}000$$

FIXED POINT: $A_0 = 43{,}333$

$B_0 = 16{,}667$

$C_0 = 100{,}000 - 43{,}333 - 16{,}667 = 40{,}000$

SINK/SOURCE/SADDLE: $\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} .5 & .1 \\ 0 & .4 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} + \begin{bmatrix} 20{,}000 \\ 10{,}000 \end{bmatrix}$

EIGENVALUES: $(.5-r)(.4-r) = 0$

$r = .5, .4 \quad \Rightarrow \quad$ SINK

$\therefore$ OVER TIME, POPULATIONS CONVERGE TO

$A_0 = 43{,}333$, $B_0 = 16{,}667$, $C_0 = 40{,}000$.

23. In the city/suburbs/rural dwellers problem discussed in this section, suppose instead that each decade 10% of city dwellers move to the suburbs and an additional 5% to rural areas; 10% of suburban dwellers move to the city and another 10% to rural areas; 5% of rural dwellers move to the city and another 5% to the suburbs; the remainder of the total population of 100,000 remain where they are. Find the steady-state in this case and determine its stability.

25. Suppose that the next choice of vehicle for those who currently drive a small car is as follows: 60% will again choose a small car, 20% a large car, and the rest an SUV. Of those who currently drive a large car 10% will next choose a small car, 50% another large car, and the rest an SUV. Of SUV drivers 10% will next choose a small car, 10% a large car, and the rest another SUV. What percent of drivers will eventually drive small cars, what percent large cars and what percent SUV's?