§4.7 Complex Eigenvalues (continued)

Last time: When $\vec{x}_{n+1} = A\vec{x}_n$ & A has complex conjugate eigenvalue

$$z = a \pm bi = r(\cos\theta \pm i \sin\theta),$$

$$\vec{x}_n = \vec{u}\, r^n \cos(n\theta) + \vec{v}\, r^n \sin(n\theta)$$

$$\left(\begin{aligned} x_n &= u_1 r^n \cos(n\theta) + v_1 r^n \sin(n\theta) \\ y_n &= u_2 r^n \cos(n\theta) + v_2 r^n \sin(n\theta) \end{aligned} \right)$$

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ex. Find the unique solution to

$$\begin{aligned} P_{n+1} &= 2P_n - Q_n \\ Q_{n+1} &= P_n + Q_n \end{aligned} \quad ; \quad \begin{aligned} P_0 &= 1000 \\ Q_0 &= 1000 \end{aligned}$$

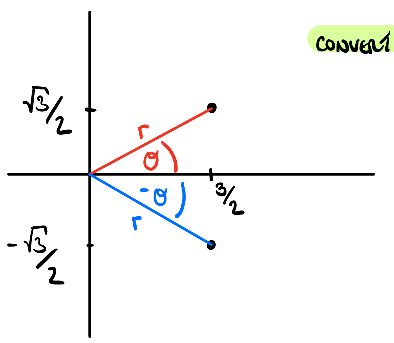
First find eigenvalues: $\vec{x}_{n+1} = A\vec{x}_n = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}_n$

$$\det(A - rI) = (2-z)(1-z) + 1 = 0$$

$$z^2 - 3z + 3 = 0 \implies z = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3 \pm \sqrt{3}\, i}{2}$$

(complex conjugates)

Convert to Polar Coord:



$$r = |z| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{3/2}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\therefore \quad \vec{x}_n = \vec{u}\, \sqrt{3}^n \cos\left(n\frac{\pi}{6}\right) + \vec{v}\, \sqrt{3}^n \sin\left(n\frac{\pi}{6}\right)$$

i.e.
$$x_n = u_1 \sqrt{3}^n \cos\left(n\frac{\pi}{6}\right) + v_1 \sqrt{3}^n \sin\left(n\frac{\pi}{6}\right)$$

$$y_n = u_2 \sqrt{3}^n \cos\left(n\frac{\pi}{6}\right) + v_2 \sqrt{3}^n \sin\left(n\frac{\pi}{6}\right)$$

$n=0$: $1000 = u_1$; Find x_1, y_1: $x_1 = 2(1000) - 1000 = 1000$

$1000 = u_2$ $\qquad y_1 = 1000 + 1000 = 2000$

$n=1$: $1000 = 1000\sqrt{3}\cos\frac{\pi}{6} + v_1 \sqrt{3} \sin\left(\frac{\pi}{6}\right)$

$$\Rightarrow 1000 = \frac{1000 \cdot 3}{2} + \frac{v_1\sqrt{3}}{2}$$

$$2000 = 3000 + \sqrt{3}\, v_1 \Rightarrow v_1 = \frac{-1000}{\sqrt{3}}$$

$$2000 = 1000\sqrt{3}\cos\left(\frac{\pi}{6}\right) + v_2\sqrt{3}\sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow 2000 = \frac{1000 \cdot 3}{2} + \frac{v_2\sqrt{3}}{2}$$

$$4000 = 3000 + \sqrt{3}\, v_2 \Rightarrow v_2 = \frac{1000}{\sqrt{3}}$$

$$\therefore \quad x_n = \sqrt{3}^n\left(1000\cos\left(n\frac{\pi}{6}\right) - \frac{1000}{\sqrt{3}}\sin\left(n\frac{\pi}{6}\right)\right)$$

$$y_n = \sqrt{3}^n\left(1000\cos\left(n\frac{\pi}{6}\right) + \frac{1000}{\sqrt{3}}\sin\left(n\frac{\pi}{6}\right)\right)$$

ROTATION & COMPLEX EIGENVALUES

COMPLEX CONJUGATE EIGENVALUES $r(\cos\theta \pm i\sin\theta)$ LEAD TO GENERAL SOLUTION

$$\vec{x}_n = \vec{u}\, r^n\cos(n\theta) + \vec{v}\, r^n\sin(n\theta) = r^n\big(\underbrace{\vec{u}\cos(n\theta) + \vec{v}\sin(n\theta)}_{\text{CALL THIS } \vec{W}_n}\big) \quad (\text{IF } r = |z| = 1)$$

n IS DISCRETE : $n = 0, 1, 2, \ldots$ BUT IF WE TEMPORARILY LET n BE CONTINUOUS, $n \in \mathbb{R}$, WE SEE

$$\vec{W}_{n+\frac{2\pi}{\theta}} = \vec{u}\cos\left(\left(n+\frac{2\pi}{\theta}\right)\theta\right) + \vec{v}\sin\left(\left(n+\frac{2\pi}{\theta}\right)\theta\right)$$

$$= \vec{u}\cos(n\theta + 2\pi) + \vec{v}\sin(n\theta + 2\pi)$$

$$= \vec{u}\cos(n\theta) + \vec{v}\sin(n\theta)$$

$$= \vec{W}_n$$

$\vec{W}_n$ WOULD BE PERIODIC FUNCTION OF n WITH PERIOD $\frac{2\pi}{\theta}$.

MOREOVER, $\vec{W}_n$ WOULD TRACE OUT A CIRCLE/ELLIPSE CENTERED AT ORIGIN.

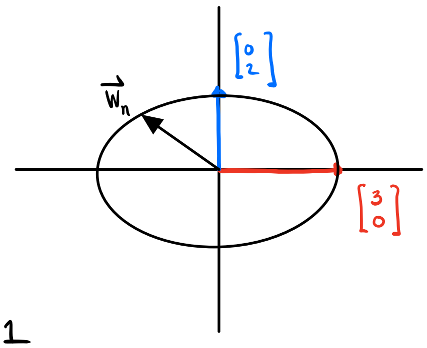
ex. Let $\vec{w}_n = \begin{bmatrix} a \\ 0 \end{bmatrix} \cos(n\theta) + \begin{bmatrix} 0 \\ b \end{bmatrix} \sin(n\theta)$

Then $x_n = a\cos(n\theta)$, $y_n = b\sin(n\theta)$

$$\Rightarrow \left(\frac{x_n}{a}\right)^2 + \left(\frac{y_n}{b}\right)^2 = 1$$

$\therefore$ (x_n, y_n) lies on ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(or circle if $a = b$).



Note: When $\vec{u}$ & $\vec{v}$ are arbitrary vectors, the generated ellipse will be rotated.

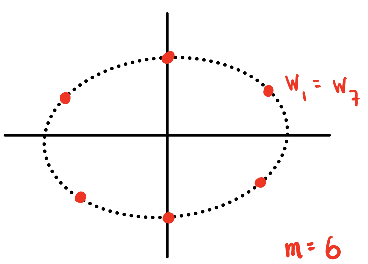
Now since n is discrete, the points described by $\vec{w}_n$ (which all lie on some ellipse) only repeat if $n\theta$: $0, \theta, 2\theta, 3\theta, \dots$ is eventually an integer multiple of 2π.

This only occurs if $\theta = \frac{p}{q}\pi$. This is usually not the case!

Here we say θ is a rational multiple of π.

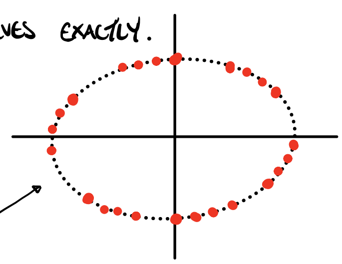
Let m be the smallest positive integer such that $m\theta = 2\pi N$ for some $N \in \mathbb{N}$.

Then solution $\vec{w}_n$ is periodic with period m.



If θ is an irrational multiple of π then the points $\vec{w}_n$ never quite repeat themselves exactly.

∞ many points,
no repeats!



We say $\vec{w}_n$ is irrationally periodic.

ex. $\vec{x}_{n+1} = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix} \vec{x}_n$.

EIGENVALUES : $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \cos\left(\frac{\pi}{3}\right) \pm i \sin\left(\frac{\pi}{3}\right)$

SINCE $|z| = 1$, SOLUTIONS LIE ON AN ELLIPSE.

THEN $\vec{x}_n = \vec{u} \cos\left(n\frac{\pi}{3}\right) + \vec{v} \sin\left(n\frac{\pi}{3}\right)$ ($\vec{u}$ & $\vec{v}$ WOULD BE DETERMINED BY INITIAL CONDITIONS)

θ = RATIONAL MULTIPLE OF π

& SINCE $6\theta = 6\left(\frac{\pi}{3}\right) = 2\pi$, SOLUTIONS HAVE PERIOD 6.

NOW CONSIDER THE GENERAL SOLUTION TO $\vec{x}_{n+1} = A\vec{x}_n$.

$\vec{x}_n = r^n \left(\vec{u}\cos(n\theta) + \vec{v}\sin(n\theta)\right) = r^n \vec{w}_n$.

PARALLEL VECTORS

IF $\vec{w}_n$ ROTATES ABOUT ORIGIN, THEN SO DOES $\vec{x}_n$!

IF $|r| > 1$, $\vec{x}_n$ SPIRALS AWAY FROM ORIGIN (SOURCE)

IF $|r| < 1$, $\vec{x}_n$ SPIRALS IN TOWARD ORIGIN (SINK)

EXAMPLE 6

Determine the nature of the trajectories in solution space for the system

$$x_{n+1} = 0.80x_n - 0.44y_n$$
$$y_{n+1} = x_n + 0.70y_n$$

Solution: The eigenvalues are $(3 \pm i\sqrt{7})/4$, which yields $r = |(3 \pm i\sqrt{7})/4| = 1$ and $\theta = \tan^{-1}(\sqrt{7}/3)$, which is an irrational multiple of π. All trajectories are elliptical around the origin and irrationally periodic, and so the points (x_n, y_n) never quite repeat themselves. Instead, they gradually trace out an entire ellipse as $n \to \infty$ (see Fig. 4.21). ■

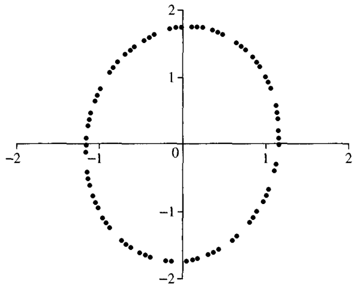


FIGURE 4.21

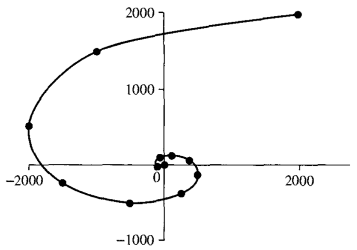


FIGURE 4.22

EXAMPLE 7

Determine the nature of trajectories in solution space for the prey–predator model

$$P_{n+1} = 0.5P_n - Q_n$$
$$Q_{n+1} = 0.25P_n + 0.5Q_n$$

Solution: This time the eigenvalues $(1 \pm i)/2$ satisfy $r = |(1 \pm i)/2| = 1/\sqrt{2}$ and $\theta = \pi/4$. The origin must be a sink and trajectories must spiral inward toward it. The angle of the solution vector is periodic with period $m = 8$, since $8(\pi/4)/(2\pi) = 1$ is an integer (see Fig. 4.22). We remark once again that the system ceases to model the true populations when either becomes negative. ■