§4.7 Complex Eigenvalues

$$\vec{x}_{n+1} = A\vec{x}_n \quad : \quad \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

EIGENVALUE(S) r SATISFY $A\vec{v} = r\vec{v}$ FOR NON-ZERO $\vec{v}$

$\iff (A - rI)\vec{v} = \vec{0}$ FOR NONZERO $\vec{v}$

$\iff (A - rI)$ IS NOT INVERTIBLE

$\iff \det(A - rI) = 0 \quad \iff (a-r)(d-r) - bc = 0$

$\iff r^2 - (a+d)r + (ad - bc) = 0$

$$r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

IF THE DISCRIMINANT IS NEGATIVE, EIGENVALUES ARE COMPLEX CONJUGATES

$$z = \alpha \pm \beta i$$

(1) $\quad z^2 - (a+d)z + ad - bc = 0$

LET $z = \alpha + \beta i$, $\bar{z} = \alpha - \beta i$ BE 2 COMPLEX EIGENVALUES.

Uncoupling the General Homogeneous System

$x_{n+1} = ax_n + by_n \xrightarrow{\text{REINDEX}} x_{n+2} = ax_{n+1} + by_{n+1}$

$\xrightarrow{\times(-d)} -dx_{n+1} = -adx_n - bdy_n$

$y_{n+1} = cx_n + dy_n \xrightarrow{\times b} by_{n+1} = bcx_n + bdy_n$

$+$

$$x_{n+2} - dx_{n+1} + \cancel{by_{n+1}} = ax_{n+1} + \cancel{by_{n+1}} - adx_n - \cancel{bdy_n} + bcx_n + \cancel{bdy_n}$$

$$x_{n+2} - dx_{n+1} = ax_{n+1} - adx_n + bcx_n$$

$$x_{n+2} = (a+d)x_{n+1} - (ad-bc)x_n \qquad (2)$$

THE VALUES $x_0, x_1, x_2, \ldots$ (ALONE!) SATISFY 2^{nd} ORDER EQ (2).
"UNCOUPLED"

NOTE: IN ORDER TO ITERATE, WE NEED x_0 & x_1.

BUT GIVEN INITIAL VALUES x_0 & y_0, WE HAVE $x_1 = ax_0 + by_0$ ✓

INTERESTING

IN FACT, SIMILAR PROCEDURE SHOWS $y_0, y_1, y_2, \ldots$ SATISF THE SAME 2^{nd} ORDER EQ (2)

$$x_{n+1} = ax_n + by_n \xrightarrow{\times c} cx_{n+1} = acx_n + bcy_n$$

$$y_{n+1} = cx_n + dy_n \xrightarrow{\text{REINDEX}} y_{n+2} = cx_{n+1} + dy_{n+1}$$

$$\xrightarrow{\times(-a)} -ay_{n+1} = -acx_n - ady_n \qquad \oplus$$

$$y_{n+2} - ay_{n+1} = bcy_n + dy_{n+1} - ady_n$$

$$y_{n+2} = (a+d)\,y_{n+1} - (ad-bc)\,y_n$$

REWRITE (2) AS $x_{n+2} - (a+d)x_{n+1} + (ad-bc)x_n = 0$ (3)

RECALL: WHEN A HAS 2 REAL EIGENVALUES r & s

WITH EIGENVECTORS $\vec{u}$ & $\vec{v}$ RESPECTIVELY

WE WRITE $\vec{x}_0$ AS THE SUM OF TWO EIGENVECTORS

$\vec{x}_0 = c_1\vec{u} + c_2\vec{v}$ SO LET'S SAY $\vec{x}_0 = \vec{u} + \vec{v}$.

(ALSO EIGENVECTORS)

THEN $\vec{x}_n = A^n\vec{x}_0 = A^n(\vec{u} + \vec{v}) = A^n\vec{u} + A^n\vec{v} = r^n\vec{u} + s^n\vec{v}$.

THAT IS, $x_n = r^n u_1' + s^n v_1'$

$y_n = r^n u_2' + s^n v_2'$

$\left(\vec{u} = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \right)$

WITH $r = z$ & $s = \bar{z}$, WE HAVE $x_n = z^n u_1' + \bar{z}^n v_1'$

$y_n = z^n u_2' + \bar{z}^n v_2'$

CHECK THAT THIS IS A SOLUTION TO (3)

NOTE: SINCE $z, \bar{z}$ ARE COMPLEX #'S, u_1', u_2', v_1', v_2' MAY ALSO BE COMPLEX #'S.

($x_n, y_n \in \mathbb{R}$ REAL!)

WE FOCUS ONLY ON x_n FOR NOW. y_n IS SIMILAR.

Note that if $x_n = z^n$ $(u_1' = 1,\ v_1 = 0)$ then (3) becomes

$$z^{n+2} - (a+d)z^{n+1} + (ad-bc)z^n = 0 \qquad (\text{divide by } z^n)$$

$$z^2 - (a+d)z + (ad-bc) = 0 \qquad (1) \text{ satisfied by eigenvalues } \checkmark$$

So $x_n = z^n$ is a solution to (3).

Similarly, $x_n = \bar{z}^n$ is also a solution to (3).

And if $x_n = z^n u_1' + \bar{z}^n v_1'$ then

$$z^{n+2}u_1' + z^{n+2}v_1' - (a+d)(z^{n+1}u_1' + z^{n+1}v_1') + (ad-bc)(z^n u_1' + \bar{z}^n v_2')$$

$$= u_1'\left(z^{n+2} - (a+d)z^{n+1} + (ad-bc)z^n\right) + v_1'\left(\bar{z}^{n+2} - (a+d)\bar{z}^{n+1} + (ad-bc)\bar{z}^n\right)$$

$$= 0u_1' + 0v_1' = 0 \quad \checkmark$$

So $x_n = z^n u_1' + \bar{z}^n v_1'$ is a solution to (3) $\forall u_1', v_1' \in \mathbb{C}$

And in fact, <u>all solutions</u> can be written in this way (we will not prove this).

If we let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\bar{z} = r(\cos\theta - i\sin\theta) = re^{i(-\theta)} \qquad (r = |z|,\ \theta = \arg(z))$$

Then $x_n = r^n(\cos(n\theta) + i\sin(n\theta))u_1' + r^n(\cos(n\theta) - i\sin(n\theta))v_1'$ (4)

In theory, this is a solution for all u_1' & $v_1' \in \mathbb{C}$.

But we require x_n to be <u>real</u> (not complex) in real world applications.

MAGIC

Let $u_1' = \frac{1}{2}u_1 - \frac{1}{2}v_1 i$

$v_1' = \frac{1}{2}u_1 + \frac{1}{2}v_1 i$ where u_1, v_1 are any real #'s.

The (4) becomes

$$x_n = r^n(\cos(n\theta) + i\sin(n\theta))\left(\tfrac{1}{2}u_1 - \tfrac{1}{2}v_1 i\right) + r^n(\cos(n\theta) - i\sin(n\theta))\left(\tfrac{1}{2}u_1 + \tfrac{1}{2}v_1 i\right)$$

$$= \frac{r^n}{2}\left\{u_1\cos(n\theta) + v_1\sin(n\theta) + u_1\cos(n\theta) + v_1\sin(n\theta)\right)$$

$$+\, i \Big(\underbrace{u_1 \sin(n\theta) - v_1 \cos(n\theta) + v_1 \cos(n\theta) - u_1 \sin(n\theta)}_{0} \Big) \Big]$$

$$\therefore \boxed{\begin{aligned} x_n &= u_1 r^n \cos(n\theta) + v_1 r^n \sin(n\theta) \\ &\text{AND, SIMILARLY,} \\ y_n &= u_2 r^n \cos(n\theta) + v_2 r^n \sin(n\theta) \end{aligned} \quad \text{WHERE } \begin{aligned} &r = |z| = |\bar{z}|, \\ &\theta = \text{ARG}(z) = -\text{ARG}(\bar{z}), \\ &u_1, v_1 \in \mathbb{R}. \end{aligned}}$$

$$\Rightarrow \vec{x}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} r^n \cos(n\theta) + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} r^n \sin(n\theta)$$

OR $\boxed{\vec{x}_n = \vec{u}\, r^n \cos(n\theta) + \vec{v}\, r^n \sin(n\theta)}$

ex. (a) FIND GENERAL FORM OF ALL REAL SOLUTIONS OF THE SYSTEM

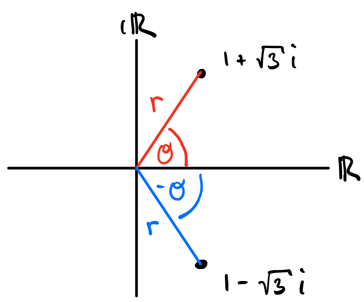
$$x_{n+1} = 2x_n - y_n$$
$$y_{n+1} = 4x_n$$

(b) FIND UNIQUE SOLUTION FOR $x_0 = 1$, $y_0 = 2$.

(a) FIND EIGENVALUES: $\det \begin{bmatrix} 2-r & -1 \\ 4 & -r \end{bmatrix} = 0 \Rightarrow (2-r)(-r) + 4 = 0$

$$r^2 - 2r + 4 = 0$$
$$r = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}\, i$$

CONVERT EIGENVALUES TO POLAR COORD.



$$r = \sqrt{1^2 + \sqrt{3}^2} = 2$$

THEN $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\therefore \boxed{\begin{aligned} x_n &= u_1 2^n \cos\left(n\tfrac{\pi}{3}\right) + v_1 2^n \sin\left(n\tfrac{\pi}{3}\right) \\ y_n &= u_2 2^n \cos\left(n\tfrac{\pi}{3}\right) + v_2 2^n \sin\left(n\tfrac{\pi}{3}\right) \end{aligned}}, \quad u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

(b) $x_0 = 1 \quad : \quad 1 = u_1$

$y_0 = 2 \quad : \quad 2 = u_2$

$$x_1 = 2x_0 - y_0 = 2(1) - (2) = 0$$

$$y_1 = 4x_0 = 4(1) = 4$$

• $x_1 = 1 \cdot 2\cos\left(\frac{\pi}{3}\right) + v_1 \cdot 2\sin\left(\frac{\pi}{3}\right) = 0$

$$\Rightarrow v_1 = \frac{-\cos(\pi/3)}{\sin(\pi/3)} = -\frac{1}{\sqrt{3}}$$

• $y_1 = 2 \cdot 2\cos\left(\frac{\pi}{3}\right) + v_2 \cdot 2\sin\left(\frac{\pi}{3}\right) = 4$

$$v_2 = \frac{2 - 2\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{2-1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

$$\therefore \begin{aligned} x_n &= u_1 2^n \cos\left(n\tfrac{\pi}{3}\right) + v_1 2^n \sin\left(n\tfrac{\pi}{3}\right) \\ y_n &= u_2 2^n \cos\left(n\tfrac{\pi}{3}\right) + v_2 2^n \sin\left(n\tfrac{\pi}{3}\right) \end{aligned} \Rightarrow \boxed{\begin{aligned} x_n &= 2^n \cos\left(n\tfrac{\pi}{3}\right) - \tfrac{1}{\sqrt{3}} 2^n \sin\left(n\tfrac{\pi}{3}\right) \\ y_n &= 2^{n+1} \cos\left(n\tfrac{\pi}{3}\right) + \tfrac{1}{\sqrt{3}} 2^{n+1} \sin\left(n\tfrac{\pi}{3}\right) \end{aligned}}$$