§ 4.4 Stability & Eigenvalues

Given any homogenous system $\begin{aligned} x_{n+1} &= a x_n + b y_n \\ y_{n+1} &= c x_n + d y_n \end{aligned}$

$$\longrightarrow \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$\boxed{\vec{x}_{n+1} = A \vec{x}_n \longrightarrow \text{Exact solution: } \vec{x}_n = A^n \vec{x}_0}$$

Does this converge or diverge?

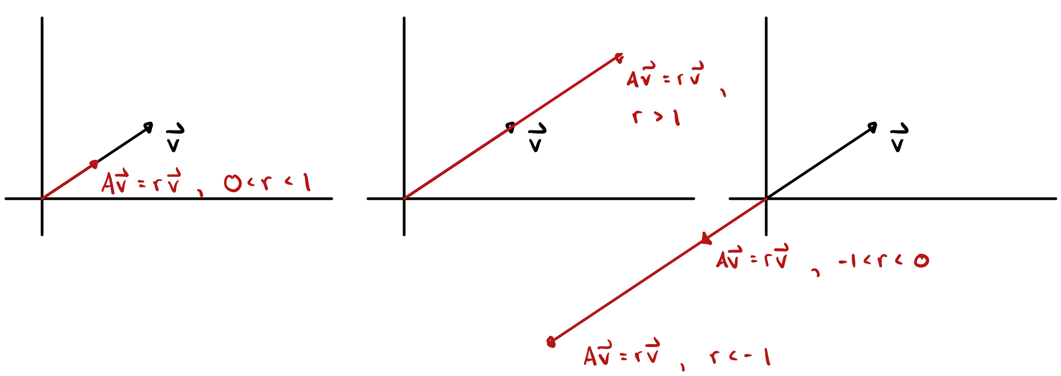
Suppose we have a matrix A

and we have a vector $\vec{v}$ & scalar r s.t.

$$A\vec{v} = r\vec{v}$$

$\vec{v}$ eigenvector

r eigen value



THEN FOR ANY CONSTANT c, $c\vec{v}$ IS ALSO AN EIGENVECTOR FOR A WITH EIGENVALUE r:

$$A(c\vec{v}) = c(A\vec{v}) = cr\vec{v} = r(c\vec{v})$$

In Exercises 1–4 verify that **u** *and* **v** *are eigenvectors of the given matrix* A *and find their corresponding eigenvalues.*

1. $A = \begin{bmatrix} -1 & 5 \\ 0.5 & 0.5 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. $A = \begin{bmatrix} 0.8 & 1.2 \\ 1 & 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
3. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$
4. $A = \begin{bmatrix} 5 & 3 \\ -2 & -0.5 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}$

SO IF $\vec{x}_{n+1} = A\vec{x}_n$ IS A HOMOGENEOUS LINEAR SYSTEM

AND $\vec{x}_0$ IS AN EIGENVECTOR OF A WITH EIGENVALUE r,

$$\vec{x}_1 = A\vec{x}_0 = r\vec{x}_0$$

$$\vec{x}_2 = A\vec{x}_1 = A(r\vec{x}_0) = r^2\vec{x}_0$$

$$\vdots$$

$$\vec{x}_n = r^n\vec{x}_0$$

THEN $|\vec{x}_n| \begin{cases} \to 0 & \text{IF } |r| < 1 \quad \text{CONVERGES} \\ \to \infty & \text{IF } |r| > 1 \quad \text{DIVERGES} \end{cases}$

(IGNORE $r = \pm 1$)

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ex. SUPPOSE

$$\begin{aligned} x_{n+1} &= x_n - y_n \\ y_{n+1} &= \frac{1}{8}x_n + \frac{1}{4}y_n \end{aligned} \quad \& \quad \vec{x}_0 = \begin{bmatrix} 8000 \\ 4000 \end{bmatrix}$$

DETERMINE THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION.

NOTE: THIS IS AN EIGENVECTOR WITH EIGENVALUE .5

WHAT IF $\vec{x}_0$ IS NOT AN EIGENVECTOR OF A?

ASSUME FOR NOW THAT A HAS 2 REAL DISTINCT EIGENVALUES r & s

& 2 EIGENVECTORS (NOT MULTIPLES OF ONE ANOTHER), $\vec{u}$ & $\vec{v}$.

$$A\vec{u} = r\vec{u} \quad , \quad A\vec{v} = s\vec{v} \quad .$$

<u>FACT</u>: THEN WE CAN FIND CONSTANTS u_1 & v_1 S.T.

$$\vec{x}_0 = u_1\vec{u} + v_1\vec{v}$$

$$\Rightarrow \quad A\vec{x}_0 = A(u_1\vec{u} + v_1\vec{v}) = A(u_1\vec{u}) + A(v_1\vec{v})$$

$$= u_1 A\vec{u} + v_1 A\vec{v}$$

$$= u_1 r\vec{u} + v_1 s\vec{v}$$

$$A^n\vec{x}_0 = u_1 r^n\vec{u} + v_1 s^n\vec{v} \quad .$$

THEOREM *Stability: Two Real Eigenvalues*

Suppose the coefficient matrix of a homogeneous linear system has two real distinct eigenvalues r and s. Then the fixed point $(0, 0)$ of the system is

(i) a sink if both $|r| < 1$ and $|s| < 1$, or
(ii) a source if both $|r| > 1$ and $|s| > 1$, or
(iii) a saddle if $|r| < 1$ and $|s| > 1$, or if $|r| > 1$ and $|s| < 1$.

NOTE: IT IS NOT NECESSARY TO KNOW THE EIGENVECTORS. KNOWING THE EIGENVALUES IS SUFFICIENT.

<u>COMPUTING EIGENVALUES</u>

(<u>DEF</u>:) r IS AN EIGENVALUE OF A IF

$A\vec{v} = r\vec{v}$ FOR SOME NONZERO VECTOR $\vec{v}$.

$\Rightarrow A\vec{v} = rI\vec{v} \quad \Rightarrow \quad A\vec{v} - rI\vec{v} = 0$

<u>SHOW MATRIX EG'S</u>

$\Rightarrow (A - rI)\vec{v} = 0$ NONTRIVIAL SOLUTION $\iff \det(A - rI) = 0$.

THAT IS : $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\right)$

$= \det\left(\begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix}\right)$

$= (a-r)(d-r) - bc = 0$

CHARACTERISTIC EQ FOR A

AND EIGEN VECTORS

In Exercises 7–10 compute the eigenvalues of the given matrix.

7. $\begin{bmatrix} 8 & -1 \\ 2 & 5 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ 9. $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ 10. $\begin{bmatrix} 1.5 & 0.5 \\ -0.5 & 0.25 \end{bmatrix}$

In Exercises 11–18, given the matrix A*, determine whether the origin is a sink, source or saddle of* $\mathbf{x}_{n+1} = A\mathbf{x}_n$.

11. The matrix A from Exercise 1.
12. The matrix A from Exercise 2.
13. The matrix A from Exercise 7.
14. The matrix A from Exercise 8.
15. $A = \begin{bmatrix} 2.25 & 2.5 \\ 0.75 & 2.0 \end{bmatrix}$
16. $A = \begin{bmatrix} 1/4 & -1/8 \\ -1/2 & 3/4 \end{bmatrix}$
17. $A = \begin{bmatrix} 0.5 & -0.75 \\ -0.25 & 0.5 \end{bmatrix}$
18. $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$

7. $(8-r)(5-r)+2 = 0$

$40 - 8r - 5r + r^2 + 2 = 0$

$r^2 - 13r + 42 = 0$

$(r-7)(r-6) = 0$

$r = 6, 7$

SOURCE

8. $(2-r)^2 - 2 = 0$

$4 - 4r + r^2 - 2 = 0$

$r^2 - 4r + 2 = 0$

$r = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$

SADDLE

$\begin{bmatrix} 8 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 6\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \longrightarrow \begin{matrix} 8u_1 - u_2 = 6u_1 \\ 2u_1 + 5u_2 = 6u_2 \end{matrix} \longrightarrow \begin{matrix} 2u_1 = u_2 \\ 2u_1 = u_2 \end{matrix}$

∞ MANY SOLUTIONS

$c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $c \in \mathbb{R}$, $c \neq 0$

$$\begin{bmatrix} 8 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - 6\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

NOT ALL 2x2 MATRICES HAVE 2 REAL EIGENVALUES

CHARACTERISTIC EQ: $\det(A - rI) = 0$

$$\det\begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix} = 0$$

$$(a-r)(d-r) - bc = 0$$

$$ad - ar - dr + r^2 - bc = 0$$

$$r^2 - (a+d)r + ad - bc = 0$$

$$r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

(1) MAY BE 0 : 1 REAL EIGENVALUE

(2) MAY BE NEG : 2 COMPLEX (CONJUGATE) EIGENVALUES

(1) **THEOREM** *Stability: One Repeated Real Eigenvalue* (§4.5)

Suppose the coefficient matrix of a homogeneous linear system has only one repeated real eigenvalue r. Then the fixed point (0, 0) of the system is:

(i) a sink if $|r| < 1$; or
(ii) a source if $|r| > 1$.

(2) **THEOREM** *Stability: Complex Conjugate Eigenvalues* (§4.7)

Suppose the coefficient matrix of a homogeneous linear system has complex conjugate eigenvalues $\alpha \pm \beta i$. Then the fixed point (0, 0) of the system is

(i) a sink if $|\alpha \pm \beta i| < 1$; or
(ii) a source if $|\alpha \pm \beta i| > 1$.