§ 4.1 Some Linear Systems Models

- Recall linear population model

$$P_{n+1} = rP_n + k \quad , \quad \begin{array}{l} r = \text{growth rate} \\ k = \text{migration / immigration / harvesting constant} \end{array}$$

* isolated species

More generally

$$x_{n+1} = f(x_n)$$

- Consider a pair of populations : Prey & Predator

$$\text{Prey} \quad P_{n+1} = r_1 P_n + k_1 - s_1 Q_n$$
$$\text{Predator} \quad Q_{n+1} = r_2 Q_n + k_2 + s_2 P_n$$

Prey/Predator interaction!

$$\text{Prey}: \quad P_{n+1} = r_1 P_n - s_1 Q_n + k_1$$
$$\text{Predator}: \quad Q_{n+1} = s_2 P_n + r_2 Q_n + k_2$$

Growth rates $r_1, r_2 \geq 0$

Proportions $s_1, s_2 \geq 0$

$k_1, k_2 \in \mathbb{R}$

- Consider a pair of competing populations :

$$P_{n+1} = r_1 P_n + k_1 - s_1 Q_n$$
$$Q_{n+1} = r_2 Q_n + k_2 - s_2 P_n$$

competition!

$$P_{n+1} = r_1 P_n - s_1 Q_n + k_1$$
$$Q_{n+1} = -s_2 P_n + r_2 Q_n + k_2$$

Growth rates $r_1, r_2 \geq 0$

Proportions $s_1, s_2 \geq 0$

$k_1, k_2 \in \mathbb{R}$

These are examples of systems of <u>linear</u> models / iterative equations

These are the type of systems we will focus on.

More generally,

$$x_{n+1} = f(x_n, y_n)$$
$$y_{n+1} = g(x_n, y_n)$$

ex. Let A & B be the populations of 2 species.

$$A_{n+1} = 2A_n + B_n - 100$$
$$B_{n+1} = -A_n + 3B_n + 25$$

Competitors? Predator/Prey?

(a) Describe the relationship between the populations & how they evolve.

(b) If $A_0 = 30$ & $B_0 = 75$, find the populations of generations 1, 2, & 3 time steps later.

Two Interacting Populations

n	P_n	Q_n	r_1	s_1	k_1
0	30	75	3	-1	25
1	40	80	s_2	r_2	k_2
2	65	100	1	2	-100
3	120	165			
4	220	350			
5	335	820			
6	210	1875			
7	-1220	3860			

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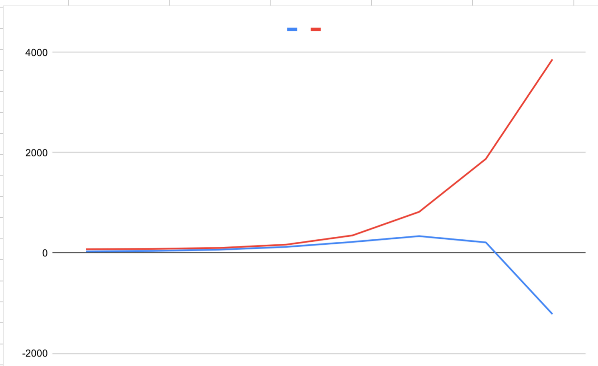


Note: Systems of linear iterative equations can be used to model any quantities that interact as they evolve. For example, price and demand.

5. Construct the linear prey–predator model with no immigration, migration or harvesting, and:
   (a) The prey's growth rate is 1.2 and the predator's is 1.3; the prey population is diminished by 0.2 times that of the predator; the predator population is increased by 0.3 times that of the prey.
   (b) On their own each population would remain fixed, but the prey population is diminished by twice the predator population, and predator population increased by twice the prey population.

8. Construct the linear competition model that satisfies:
   (a) The growth rate of the first species is 1.5, and of the second is 1.25; each is diminished by 0.4 times the population of the other.
   (b) In addition to the characteristics of (a), the first species undergoes immigration at 2500 per step, and the second migrates at 1200 per step.

Overlapping Generations Model

Suppose $P_{n+1} = rP_n + sP_{n-1} + K \qquad (r, s > 0)$

Linear System: $P_{n+1} = rP_n + sQ_n + K$

$Q_n = P_{n-1}$

$Q_{n+1} = P_n$

14. Construct the linear overlapping-generations model that satisfies:
    (a) The population of the next generation equals 3/4 of the present population, plus 1/2 of the previous population.
    (b) In addition to that described in (a), there is migration at 2500 per step.

& convert to linear system

§4.2 Linear Systems & Their Dynamics

Systems of Linear Iterative Eq's

$$x_{n+1} = a x_n + b y_n + h$$
$$y_{n+1} = c x_n + d y_n + k$$

Def: The system is HOMOGENEOUS if $h = k = 0$.

Fixed Points

DEFINITION

A point (p, q) is a **fixed point** of the linear system (1) if it satisfies

$$\begin{aligned} p &= a\,p + b\,q + h \\ q &= c\,p + d\,q + k \end{aligned} \qquad (8)$$

Letting $(x_0, y_0) = (p, q)$ will make $(x_n, y_n) = (p, q)$ for all $n \geq 0$, and so (p, q) is an **equilibrium point** of the model.

ex. Find the fixed point of the linear system

$$P_{n+1} = 3P_n - Q_n + 25$$
$$Q_{n+1} = P_n + 2Q_n - 100$$

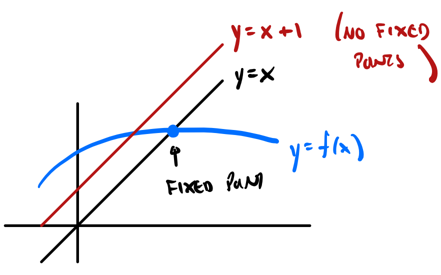
$p = 3p - q + 25 \Rightarrow 2p - q = -25$

$q = p + 2q - 100 \Rightarrow p + q = 100$

$3p = 75 \Rightarrow p = 25,\ q = 75$

Q: Do all linear systems have a unique fixed point?

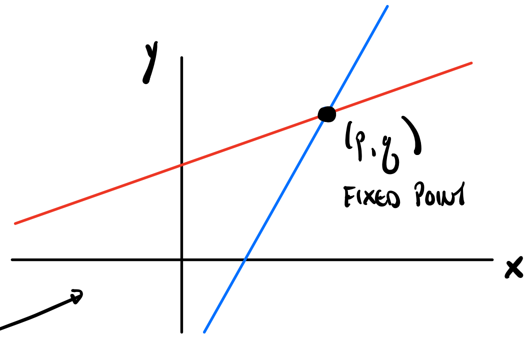
No: $x_{n+1} = x_n + 1$ ← No fixed point.



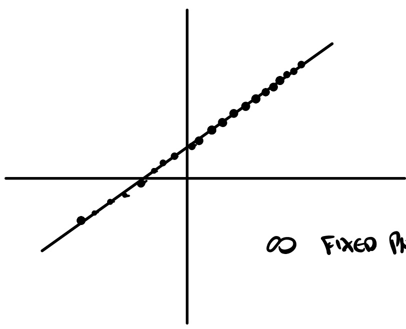
$$x_{n+1} = a x_n + b y_n + h \qquad\qquad p = ap + bq + h$$
$$y_{n+1} = c x_n + d y_n + k \qquad\qquad q = cp + dq + k$$

$\xrightarrow{\text{FIXED PT}}$ EQ'S

$$\longrightarrow \quad (a-1)p + bq = -h$$
$$cp + (d-1)q = -k$$



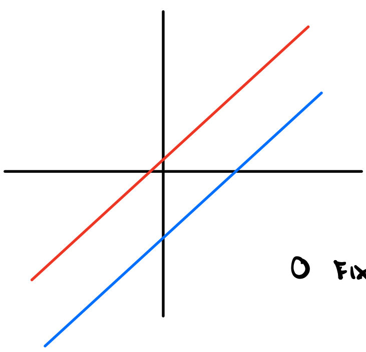
INTERSECTION IS FIXED POINT.



∞ FIXED PTS

$$(a-1)p + bq = -h$$
$$cp + (d-1)q = -k$$

IF THESE EQ ARE EQUIVALENT

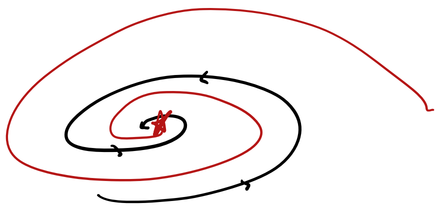


$$(a-1)p + bq = -h$$
$$cp + (d-1)q = -k$$

IF THESE EQ ARE INCONSISTANT

0 FIXED POINTS.

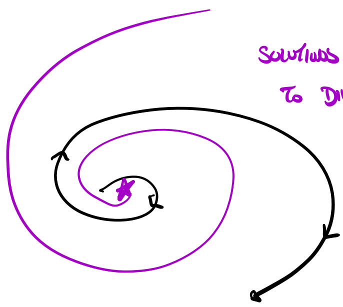
WE WILL ONLY LOOK AT SYSTEMS WITH ONE (UNIQUE) FIXED POINT.



SOLUTIONS APPEAR TO CONVERGE!

(ATTRACTOR)

* STABLE FIXED POINT OF A LINEAR SYSTEM IS CALLED A **SINK**

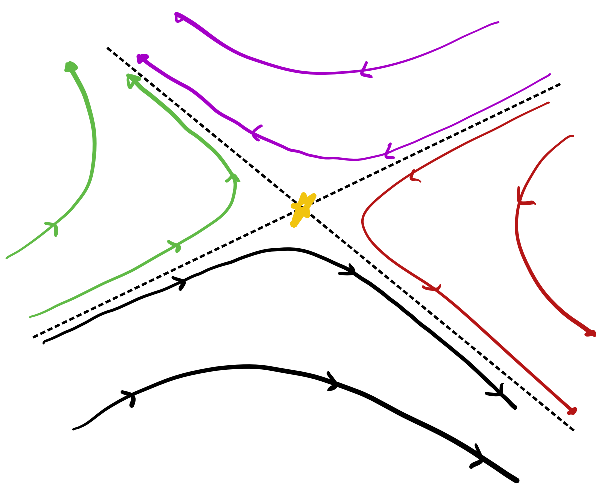


SOLUTIONS APPEAR TO DIVERGE!

(REPELLER)

* ONE TYPE OF UNSTABLE FIXED POINT OF A LINEAR SYSTEM IS CALLED A **SOURCE**

(SINK OF INVERSE SYSTEM)



HYPERBOLIC TRAJECTORIES

* THE OTHER TYPE OF UNSTABLE FIXED POINT IS NEITHER A SINK OR A SOURCE.

SOLUTIONS DIVERGE FROM ONE LINE & CONVERGE TO ANOTHER.

THE INTERSECTION OF THESE LINES IS A HYPERBOLIC FIXED POINT, OR A SADDLE POINT.

For each of the systems given in Exercises 11 and 12 explain why a unique fixed point does not exist.

11. $x_{n+1} = 2x_n - y_n$
$y_{n+1} = x_n$

12. $x_{n+1} = 2x_n + y_n + 2$
$y_{n+1} = 3x_n + 4y_n + 9$

In Exercises 13–16 find a formula for the exact solution of the given system.

13. $x_{n+1} = a\,x_n$
$y_{n+1} = d\,y_n$

14. $x_{n+1} = ax_n$
$y_{n+1} = cx_n$

15. $x_{n+1} = a\,x_n + h$
$y_{n+1} = d\,y_n + k$

16. $x_{n+1} = a\,x_n + h$
$y_{n+1} = cx_n + k$

In Exercises 17–20 find the fixed point of the given system and determine whether it is a sink, source or saddle using the results of Exercises 13–16.

17. $x_{n+1} = \frac{2}{3}x_n + 40$
$y_{n+1} = 2x_n - 80$

18. $x_{n+1} = 2.5x_n$
$y_{n+1} = 0.8y_n + 2000$

19. $x_{n+1} = \frac{3}{2}x_n - 2500$
$y_{n+1} = \frac{3}{4}y_n + 700$

20. $x_{n+1} = \frac{1}{3}y_n - 10$
$y_{n+1} = 2y_n + 5$

16. $X_n = a^n X_0 + \frac{1-a^n}{1-a}h$

$y_n = c\left(a^n X_0 + \frac{1-a^n}{1-a}h\right) + k$