§3.6 Parameterized Families

Logistic Model $\quad P_{n+1} = rP_n\left(1 - \frac{P_n}{C}\right)$

r, C constants called parameters

A particular logistic model depends on choice of param r, C

We would like to study ALL logistic models at once!

Change of variable ⟶ One-Parameter Family

$P_{n+1} = rP_n\left(1 - \frac{P_n}{C}\right) \qquad$ let $\boxed{P_n = Cx_n}$. $P_{n+1} = Cx_{n+1}$

$\cancel{C}x_{n+1} = r\cancel{C}x_n\left(1 - \frac{\cancel{C}x_n}{\cancel{C}}\right) \longmapsto \boxed{x_{n+1} = rx_n(1 - x_n)}$

One-parameter family of functions.

Only 1-parameter!

C : scaling factor.

$$\boxed{\begin{aligned} &P_{n+1} = rP_n e^{-P_n/N}, \quad P_n = Nx_n \\ &\cancel{N}x_{n+1} = r\cancel{N}x_n e^{-\cancel{N}x_n/\cancel{N}} \Rightarrow x_{n+1} = rx_n e^{-x_n} \end{aligned}}$$

Another one-param family

e.g. If p is a fixed point of $x_{n+1} = rx_n(1 - x_n)$

then Cp is a fixed point of $P_{n+1} = rP_n\left(1 - \frac{P_n}{C}\right)$

$f_2(x) = 2x(1-x)$

Logistic Family: $f_r(x) = rx(1-x)$

f depends on r

How do the dynamics depend on r?

Does changing r change fixed points?
# of fixed points?
# of m-cycles?

Stability?

ex. Find all non-negative fixed points of logistic family & determine their stability for $r > 0$.

$$f_r(x) = rx(1-x) = x$$

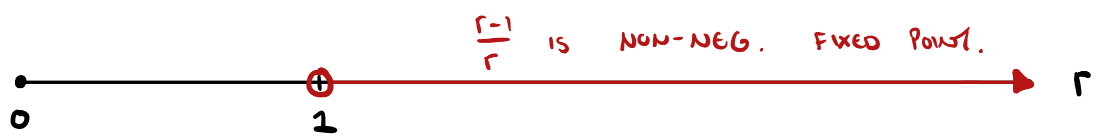
$$x\left[r(1-x) - 1\right] = 0 \qquad x = 0 \text{ fixed point}$$

$$r - rx - 1 = 0$$

Another non-neg. fixed point

$$rx = r - 1 \qquad p(r) = \frac{r-1}{r}, \quad \underbrace{r > 1}_{\text{interval of existence}}$$

$$x = \frac{r-1}{r}$$



$x = 0$ is always a non-neg. fixed point

$$f_r(x) = rx(1-x) = rx - rx^2$$

$$f_r'(x) = r - 2rx = r(1-2x)$$

$f_r'(0) = r \implies 0$ is a stable fixed point for $0 < r < 1$
unstable for $r > 1$

$$f_r'(p(r)) = f_r'\left(\frac{r-1}{r}\right) = r\left(1 - 2\left(\frac{r-1}{r}\right)\right) = r - 2(r-1) = 2 - r$$

$$r - 2(r-1) = r - 2r + 2$$

$\therefore \; p(r) = \frac{r-1}{r}$ IS STABLE IF $|2-r| < 1$

$$-1 < 2 - r < 1$$

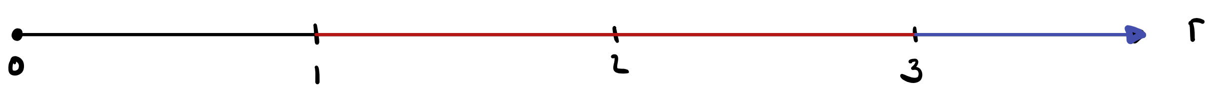
$$-3 < -r < -1$$

$$1 < r < 3$$

$p(r) = \frac{r-1}{r}$ IS UNSTABLE IF $r > 3$

STABLE FIXED POINT $p(r) = \frac{r-1}{r}$

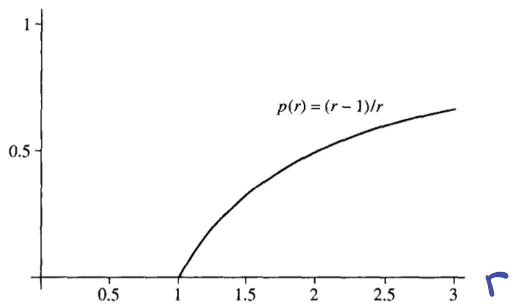
UNSTABLE FIXED POINT $p(r)$



INTERVAL OF STABILITY : $p(r) = \frac{r-1}{r}$ IS STABLE IF $1 < r < 3$

INTERVAL OF EXISTENCE

$p(r) = \frac{r-1}{r}$ IS FIXED FOR $r > 1$



POSITIVE FIXED POINT OF f_r

DEFINITION

The interval of parameter values for which a fixed or periodic point exists for a parameterized family is called the **interval of existence** for the point. The interval for which such a point is stable is called its **interval of stability.**

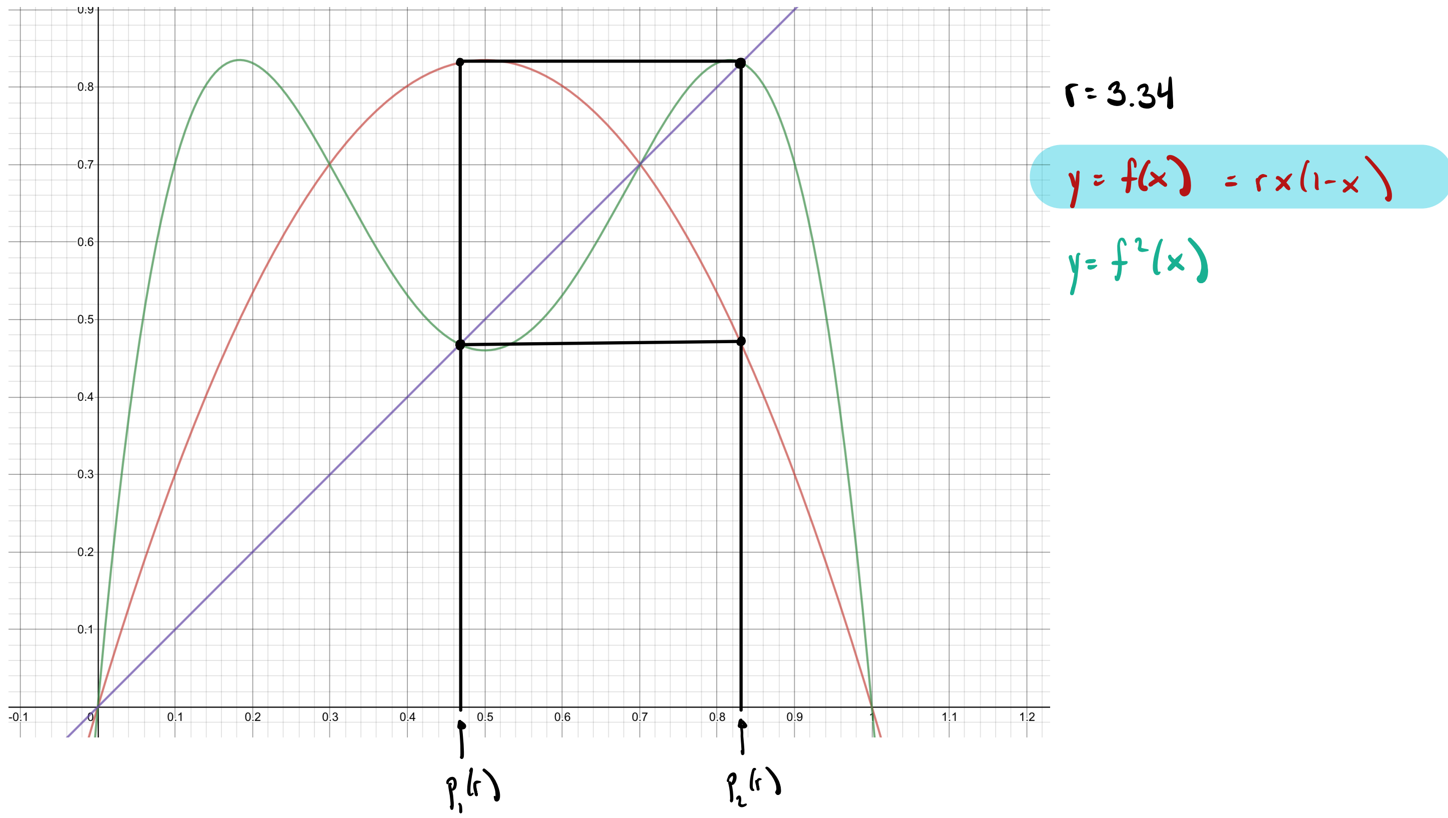
HOW USEFUL IS YOUR MODEL ?

OUR CHOICE OF r MAY BE INACCURATE.

BUT IF THE INTERVAL OF EXISTENCE / STABILITY IS LARGE THIS WON'T ALTER THE DYNAMICS.

EXAMPLE 5

(a) Find the 2-cycle of the logistic family (2) for $r > 0$. (b) Determine its intervals of existence and stability.



FIND 2-CYCLE: FIXED POINTS OF $f_r(f_r(x)) = x$

$$r[rx(1-x)](1-[rx(1-x)]) = x$$

$$r^3x^4 - 2r^3x^3 + (r^3+r^2)x^2 + (1-r^2)x = 0 \qquad \div r^3$$

$$(*) \qquad x^4 - 2x^3 + \frac{r+1}{r}x^2 + \frac{1-r^2}{r^3}x = 0$$

SOLUTION TO THIS EQ ARE FIXED PTS OF $f_r^2(x)$.

$\underbrace{\text{FIXED POINTS OF } f_r(x)}$ ARE ALSO FIXED POINTS OF $f_r^2(x)$.

$$rx(1-x) = x \implies rx(1-x) - x = 0 \implies x[r(1-x)-1] = 0$$

$$\implies x = 0, \qquad r - 1 - rx = 0 \implies x = \frac{r-1}{r}$$

$\therefore$ $x = 0$ AND $x = \dfrac{r-1}{r}$ ARE ROOTS OF $(*)$

$$(*) \qquad x^4 - 2x^3 + \frac{r+1}{r}x^2 + \frac{1-r^2}{r^3}x = 0$$

$$x\left(x^3 - 2x^2 + \frac{r+1}{r}x + \frac{1-r^2}{r^3}\right) = 0$$

$$x\left(x - \frac{r-1}{r}\right)\left[x^2 - \frac{r+1}{r}x + \frac{r+1}{r^2}\right] = 0$$

$$x - \frac{r-1}{r} \ \overline{\Big)\, x^3 - 2x^2 + \frac{r+1}{r}x + \frac{1-r^2}{r^3}} \quad \text{quotient: } x^2 - \frac{r+1}{r}x + \frac{r+1}{r^2}$$

$$-\left(x^3 - \frac{r-1}{r}x^2\right)$$

$$\frac{-2r+r-1}{r}$$

$$\frac{-r-1}{r}x^2 + \frac{r+1}{r}x + \frac{1-r^2}{r^3}$$

$$-\left(-\frac{r+1}{r}x^2 + \frac{r^2-1}{r^2}x\right)$$

$$\cancel{\left(\frac{r^2+r}{r^2} - \frac{r^2-1}{r^2}\right)x + \frac{1-r^2}{r^3}}$$

$$\frac{r+1}{r^2}x + \frac{1-r^2}{r^3}$$

$$-\left(\frac{r+1}{r^2}x - \frac{r^2-1}{r^3}\right)$$

$$0 \qquad 0$$

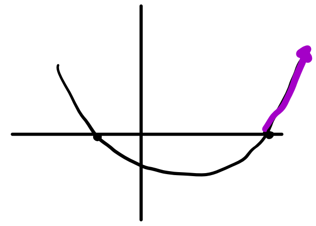
REMAINDER 0

$$\underset{\checkmark}{x}\underset{\checkmark}{\left(x - \frac{r-1}{r}\right)}\underbrace{\left[x^2 - \frac{r+1}{r}x + \frac{r+1}{r^2}\right]} = 0$$

$$x = \frac{\frac{r+1}{r} \pm \sqrt{\left(\frac{r+1}{r}\right)^2 - 4\left(\frac{r+1}{r^2}\right)}}{2}$$

$$p_1(r) = \frac{r+1-\sqrt{(r+1)(r-3)}}{2r} \qquad p_2(r) = \frac{r+1+\sqrt{(r+1)(r-3)}}{2r}$$

THESE POINTS OF PERIOD 2 EXIST WHEN $\underbrace{(r+1)(r-3) > 0}_{r<-1 \text{ or } r>3}$



ASSUME $r > 0$

INTERVAL OF EXISTENCE FOR 2-CYCLE : $r > 3$

$$p_1(r) = \frac{r+1-\sqrt{(r+1)(r-3)}}{2r}$$

$$p_2(r) = \frac{r+1+\sqrt{(r+1)(r-3)}}{2r}$$

INTERVAL OF STABILITY : $\left| f_r'(p_1(r))\, f_r'(p_2(r)) \right| < 1$

$$\Rightarrow \quad 1-\sqrt{6} < r < 1+\sqrt{6}$$

$$\Rightarrow \quad 3 < r < 1+\sqrt{6}$$

ex. FIND ALL POSITIVE FIXED POINTS & THEIR INTERVALS OF EXISTENCE & STABILITY FOR $a > 0$:

$$f_a(x) = ax - x^4$$

FIXED POINT(S) $\quad f_a(x) = x$

$$ax - x^4 = x$$

$$(a-1)x - x^4 = 0$$

$$x\left[(a-1) - x^3\right] = 0$$

$x = 0$ $\qquad$ $x^3 = a - 1$

$x = (a-1)^{1/3} > 0$ $\qquad$ POSITIVE FIXED POINT.

$a - 1 > 0$

$a > 1$

INTERVAL OF EXISTENCE: $\exists$ A POSITIVE FIXED POINT $p_a = (a-1)^{1/3}$ FOR $a > 1$

FOR WHAT VALUES OF a IS $p_a = (a-1)^{1/3}$ A STABLE FIXED POINT?

STABLE IF $|f_a'(p_a)| < 1$, UNSTABLE OTHERWISE.

$f_a(x) = ax - x^4$

$f_a'(x) = a - 4x^3 \Rightarrow f_a'(p_a) = a - 4p_a^3 = a - 4\left[(a-1)^{1/3}\right]^3$

$= a - 4(a-1) = a - 4a + 4 = 4 - 3a$

$-1 < 4 - 3a < 1$

$-5 < -3a < -3$

$\frac{-5}{-3} > a > \frac{-3}{-3} \Rightarrow 1 < a < \frac{5}{3}$ INT. OF STABILITY.

INTERVAL OF EXISTENCE: $\exists$ A POSITIVE FIXED POINT $p_a = (a-1)^{1/3}$ FOR $a > 1$

INTERVAL OF STABILITY: FIXED POINT $p_a = (a-1)^{1/3}$ IS STABLE FOR $1 < a < \frac{5}{3}$