§3.4 Some Mathematical Applications

Newton's Method for Root Finding

Q: Find the solution to $\underbrace{x^5 + x^3 + x} = 2$

$f(x) = x^5 + x^3 + x = 2$, continuous.

$$\left.\begin{aligned} f(0) &= 0 \\ f(1) &= 3 \end{aligned}\right\}$$ Intermediate Value Thm $\Rightarrow$ $\exists\, c \in (0,1)$ s.t. $f(c) = 2$.

There is at least one sol'n!

$f'(x) = 5x^4 + 3x^2 + 1 \geq 1 \Rightarrow f$ is increasing on $(-\infty, \infty)$.

There is at most one sol'n!

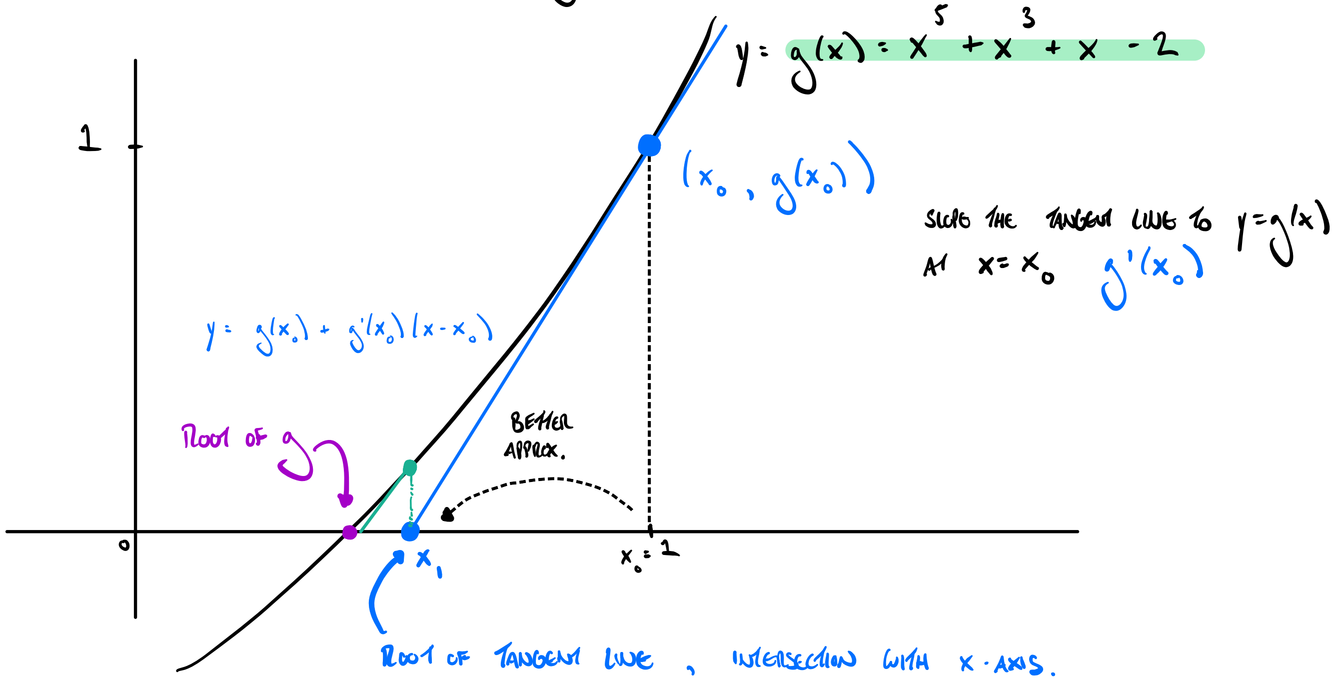
$\therefore$ A unique sol'n exists!

Exact solution may be impossible!

Approximate solution:

Rephrase: $f(x) = x^5 + x^3 + x = 2$

Instead solve $g(x) = f(x) - 2 = x^5 + x^3 + x - 2 = 0$



1. GUESS : $g(1) = 1$ CLOSE TO 0. $\longrightarrow x_0 = 1$ (GUESS)

2. EQ OF TANGENT LINE TO $y = g(x)$ AT $(x_0, g(x_0))$

$$y = \underbrace{g(x_0) + g'(x_0)(x - x_0) = 0}$$

FIND WHERE TANGENT LINE INTERSECTS THE X-AXIS

← CALL THIS x_1

$$g'(x_0)(x - x_0) = -g(x_0)$$

$$x - x_0 = -\frac{g(x_0)}{g'(x_0)}$$

$$x = x_0 - \frac{g(x_0)}{g'(x_0)}$$

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

REPEAT (ITERATE)

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}$$

NEWTON'S METHOD OF ROOT-FINDING: $x_{n+1} = f(x_n) = x_n - \dfrac{g(x_n)}{g'(x_n)}$

START WITH x_0 = BEST GUESS FOR SOLUTION TO $g(x) = 0$ (ROOT)

x_1 = BETTER GUESS

$\vdots$

$$\lim_{n \to \infty} x_n = \text{ROOT} \quad \text{(HOPEFULLY!)}$$

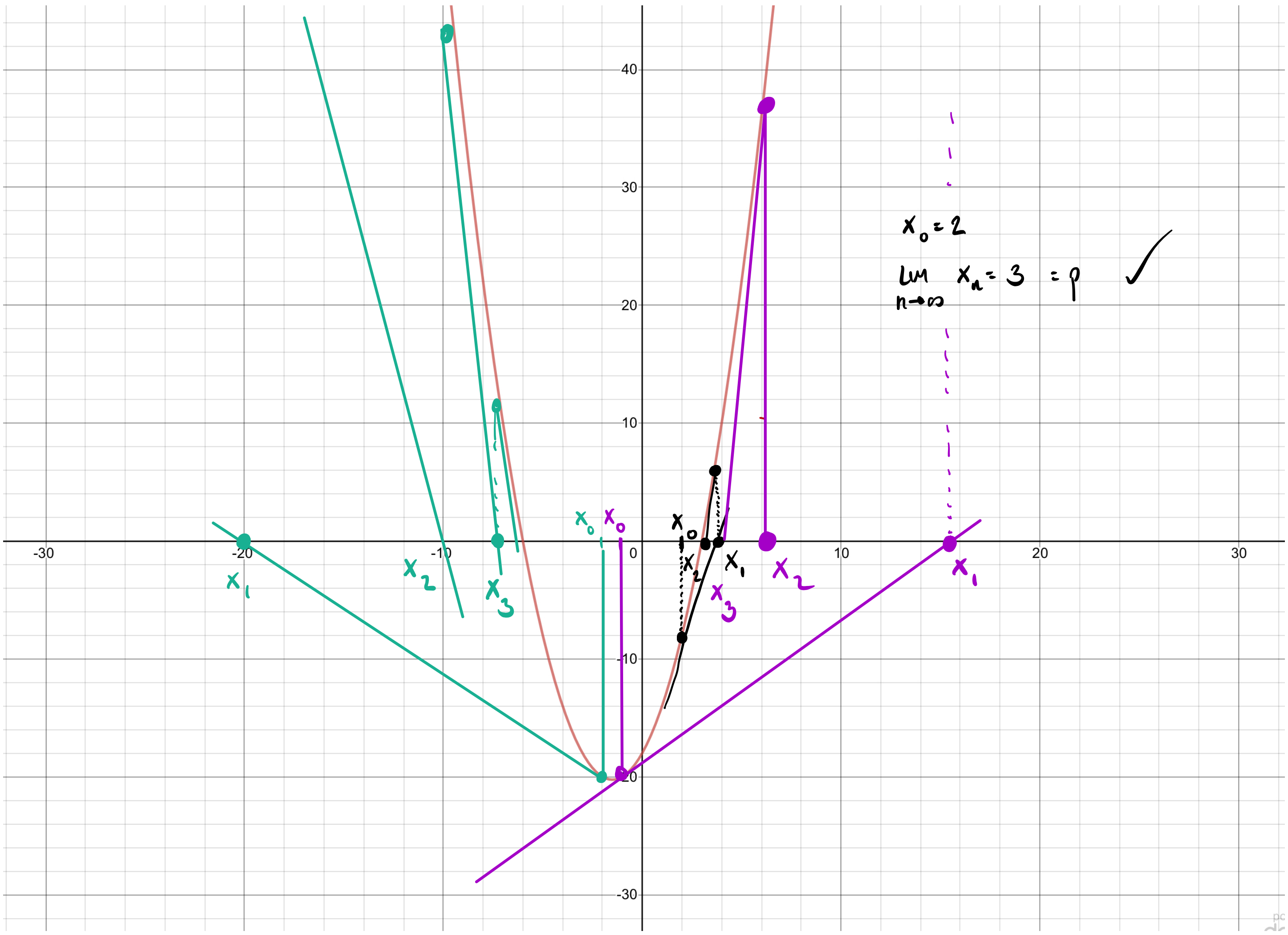
NEWTON ROOT-FINDING METHOD

To find a root p of $g(x) = 0$, if x_0 is any initial guess sufficiently close to p, then

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \tag{1}$$

will generate increasingly better approximations, and $\lim_{n\to\infty} x_n = p$.

7. The function $g(x) = x^2 + 3x - 18$ has a root at $p = 3$. Determine graphically which of the following values of x_0 will generate a sequence x_n that converges to p using the Newton method. (a) $x_0 = 2$; (b) $x_0 = -1$; (c) $x_0 = -2$.



ex. USE NEWTON'S METHOD TO APPROXIMATE $\tan^{-1}(a)$.

SOLVE $g(x) = 0$

MAKE THIS THE ROOT OF A FUNTION g.

$\tan^{-1}(a) = x$ FIND x

$a = \tan x$ FIND x

$\boxed{0 = \tan x - a}$ SOLVE FOR x.

SET $g(x) = \tan x - a$. FIND x S.T. $g(x) = 0$.

$g'(x) = \sec^2 x$

(ex. SET $a = 1$)

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \quad x_0 = 1 \in (0, \pi/2)$$

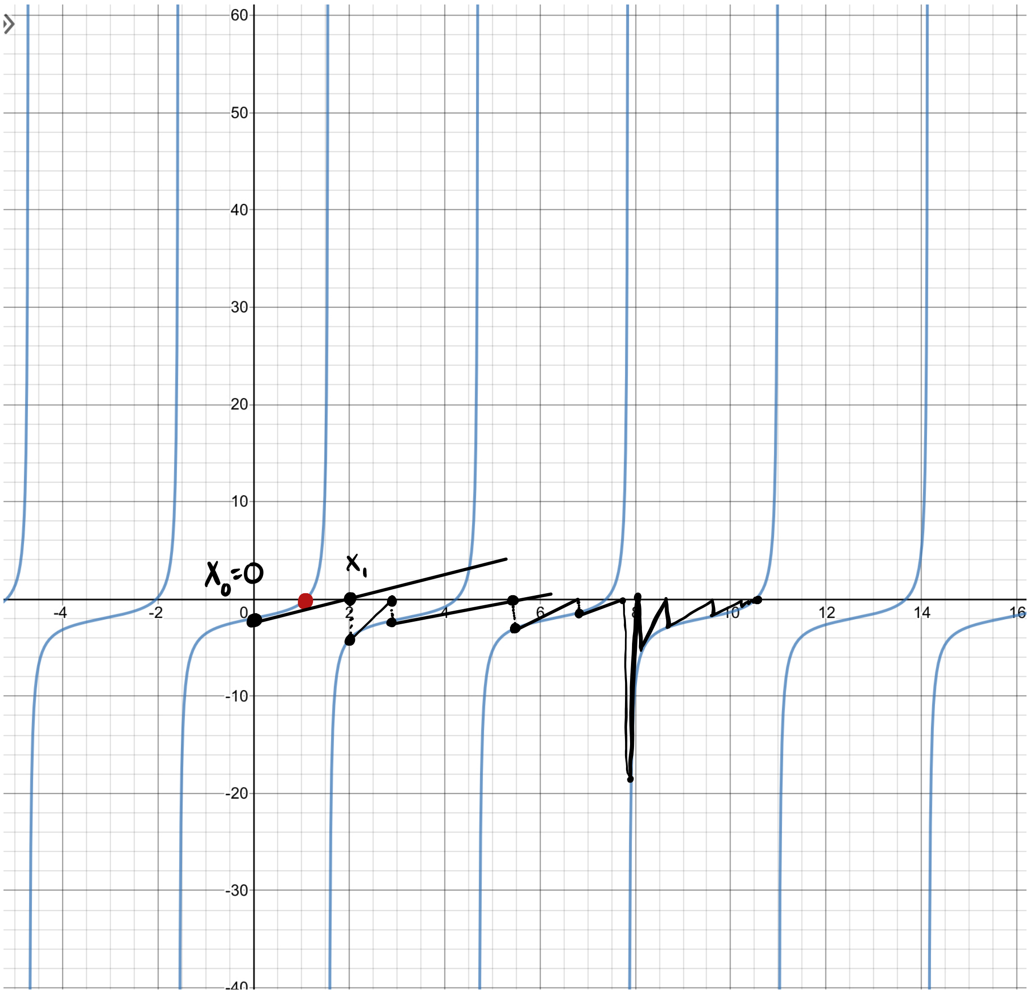
$$x_1 = 1 - \frac{\tan(1) - 1}{\sec^2(1)}$$

$$x_2 = x_1 - \frac{\tan(x_1) - 1}{\sec^2(x_1)}$$

$$x_3 = x_2 - \frac{\tan(x_2) - 1}{\sec^2(x_2)}$$

$$\vdots$$

$$\lim_{n \to \infty} x_n = \pi/4.$$



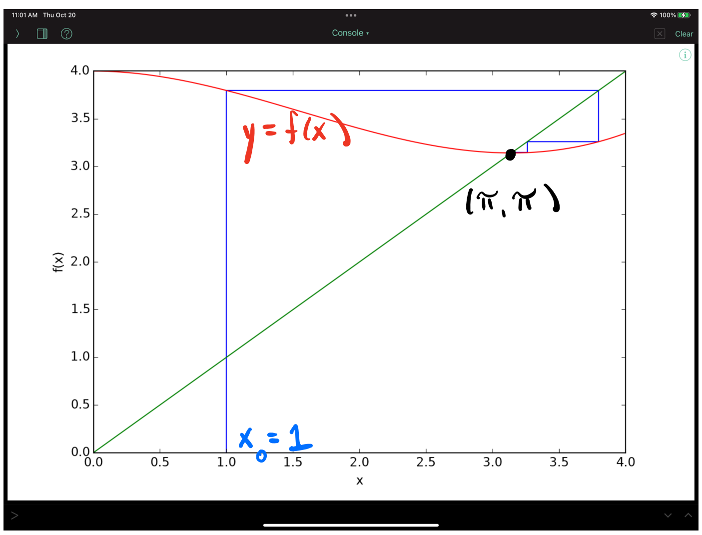
$y = \tan(x) - 2$

ALT: To approximate π:

π is a root of $g(x) = \tan\left(\frac{x}{4}\right) - 1$

Newton's method $\Rightarrow$ π is stable fixed point of

$$f(x) = x - \frac{\tan\left(\frac{x}{4}\right) - 1}{\frac{1}{4}\sec^2\left(\frac{x}{4}\right)}$$



$x_0 \rightarrow$

$x_5 \approx \pi \rightarrow$

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1
3.79631404657
3.25943543618
3.14513155421
3.14159578639
3.14159265359
3.14159265359
3.14159265359
3.14159265359
3.14159265359
3.14159265359
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NEWTON ROOT-FINDING METHOD

To find a root p of $g(x) = 0$, if x_0 is any initial guess sufficiently close to p, then

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \tag{1}$$

will generate increasingly better approximations, and $\lim_{n\to\infty} x_n = p$.

PROOF: WE NEED TO SHOW THAT THE ROOT p IS A STABLE FIXED POINT OF $x_{n+1} = f(x_n) = x_n - \dfrac{g(x_n)}{g'(x_n)}$

IF $g(x) = 0$, x IS A ROOT OF g

IF $g(x) = 0$, $g'(x) = 0$, $g''(x) = 0$, ..., $g^{(k-1)}(x) = 0$, $g^{(k)}(x) \neq 0$

THEN x IS A ROOT OF MULTIPLICITY k OF g.

MULTIPLICITY 1 : SIMPLE ROOT.

* (ASSUME ROOT OF g IS A SIMPLE ROOT : $g'(p) \neq 0$.)

FIXED POINT: SHOW THAT THE ROOT OF g IS A FIXED POINT OF

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

LET $g(p) = 0$. $\Rightarrow$ $p - \dfrac{g(p)}{g'(p)} = p - \dfrac{0}{g'(p)} = p$ ✓

STABLE: $\dfrac{d}{dx}\left[x - \dfrac{g(x)}{g'(x)} \right] = 1 - \dfrac{(g'(x))^2 - g(x)g''(x)}{(g'(x))^2}$

$$f'(p) = 1 - \frac{(g'(p))^2 - g(p)\,g''(p)}{(g'(p))^2}$$

$$= 1 - \frac{g'(p)^2}{g'(p)^2} = 1 - 1 = 0$$

p IS STABLE FIXED PNT $\because$ $|f'(p)| < 1$.

$\blacksquare$