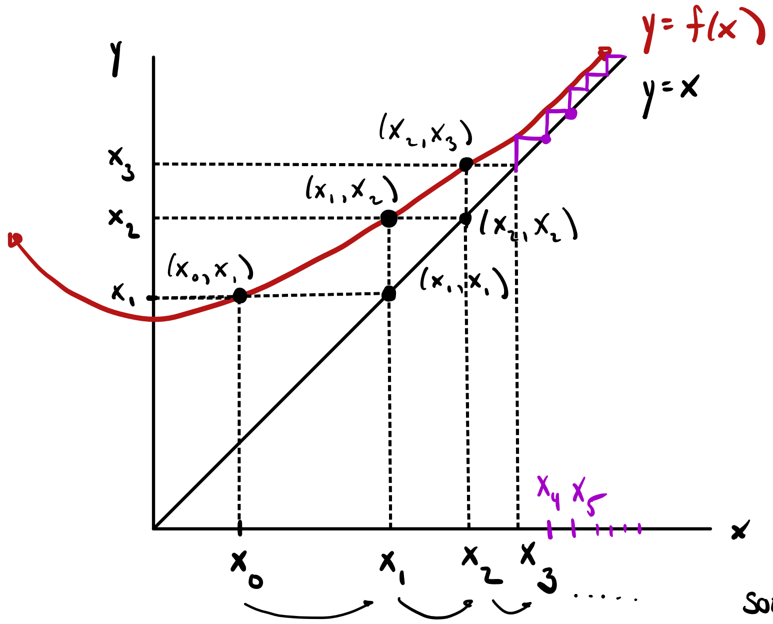
# §3.3 Cobwebbing, Derivatives, & Dynamics

Cobwebbing is a graphical method for quickly studying the dynamics of a system $x_{n+1} = f(x_n)$.

The graph $y = f(x)$ can help us determine the nature of a solution (dynamics)




$x_{n+1} = f(x_n)$

$x_1 = f(x_0)$

$x_2 = f(x_1)$

Solution is increasing

Remember to move Vertically to Graph &

Horizontally to Diagonal

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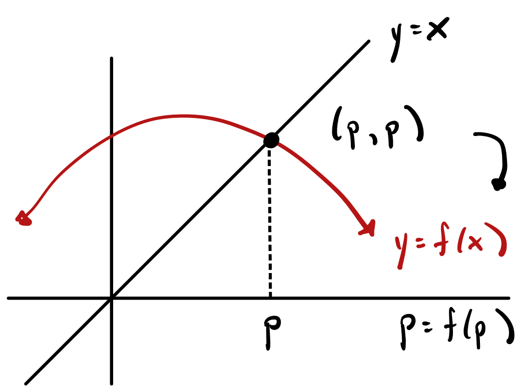
Fixed points of $x_{n+1} = f(x_n)$ are x-coord of intersections of $\underbrace{y = f(x) \ \& \ y = x}$.

$f(x) = x$

Fixed Point EQ ✓

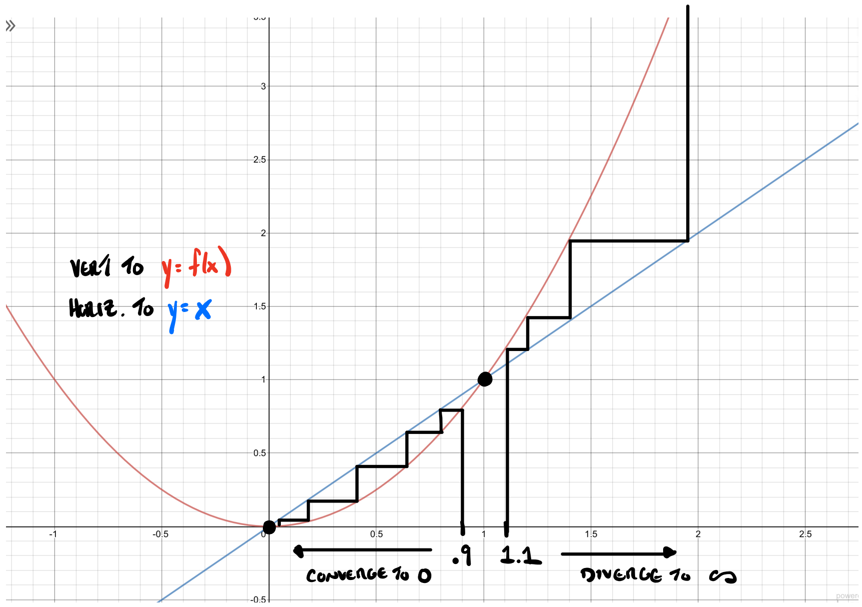


ex. DRAW A COBWEB GRAPH FOR $X_{n+1} = x_n^2$

$p = f(p)$ SOLVE FOR $p$

(a) STARTING AT 0.9

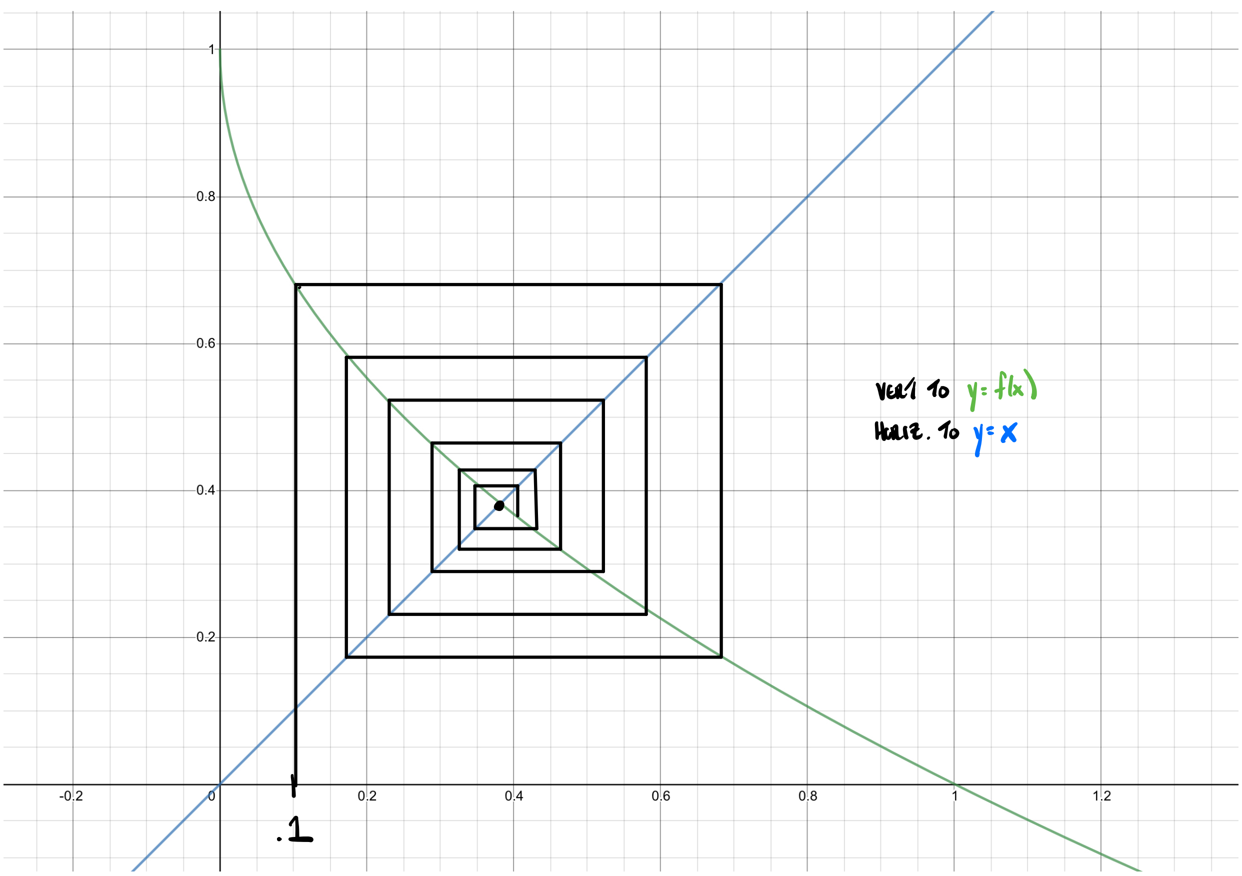
(b) STARTING AT 1.1.




FIXED POINTS : INTERSECTIONS OF $y=f(x)$, $y=x$

**ex.** DRAW A COBWEB GRAPH FOR $X_{n+1} = 1 - \sqrt{X_n}$

STARTING AT 0.1



★ COBWEBS ALLOW US TO QUICKLY FIND FIXED POINTS & CLASSIFY THEM AS EITHER STABLE OR UNSTABLE.

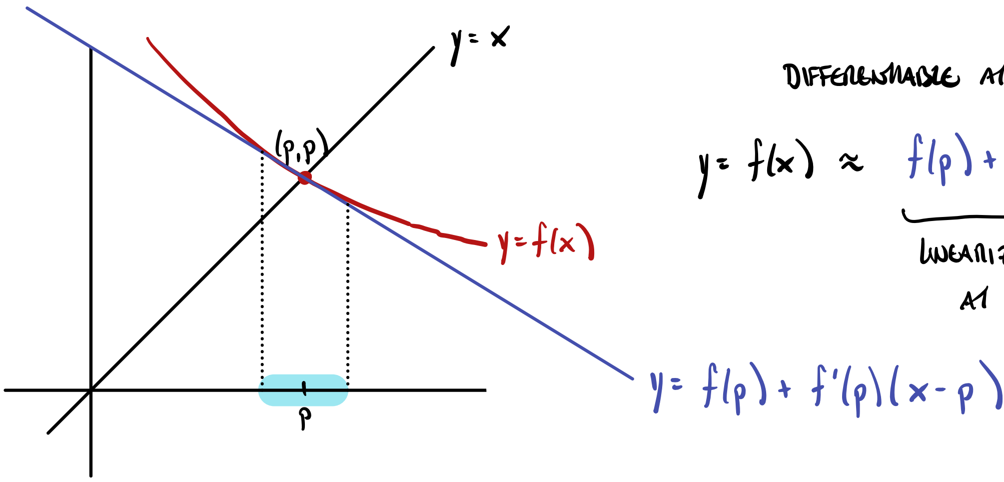
**Cobweb Dynamics**

For the equation $x_{n+1} = f(x_n)$, any point $p$ where the graphs of $y = f(x)$ and $y = x$ intersect is a fixed point. If all cobwebs starting close to $p$ converge to $(p, p)$, then $p$ is stable. If a cobweb diverges from $(p, p)$, then $p$ is unstable. And a spiral cobweb around $(p, p)$ indicates that solutions oscillate around $p$.

## DERIVATIVES (& FIXED POINTS)

NOW SUPPOSE $p$ IS A FIXED POINT OF $f(x)$ $\left( p = f(p) \right)$

& $f$ IS DIFFERENTIABLE AT $p$ $\left( f'(p) \text{ EXISTS} \right)$.




DIFFERENTIABLE AT $p$:

$$y = f(x) \approx \underbrace{f(p) + f'(p)(x-p)}_{\text{LINEARIZATION OF } f \text{ AT } p}$$

WHEN $x$ IS CLOSE TO $p$

## STABILITY & OSCILLATION THEOREM

NOW SUPPOSE $p$ IS A FIXED POINT OF $x_{n+1} = f(x_n)$, AND $f$ IS DIFFERENTIABLE AT $p$ AND $f'$ IS CONTINUOUS AT $p$.

LOCALLY, ON A SUFFICIENTLY SMALL INTERVAL CENTERED AT $p$,

$$f(x) \approx f'(p)\, x + p\left(1 - f'(p)\right)$$

IS APPROXIMATELY LINEAR WITH SLOPE $f'(p)$

THEREFORE

$$\boxed{\begin{array}{l} \text{IF } |f'(p)| > 1 \text{ THEN } p \text{ IS UNSTABLE} \\ \text{IF } |f'(p)| < 1 \text{ THEN } p \text{ IS STABLE} \\ \text{SOLUTIONS OSCILLATE IF \& ONLY IF } f'(p) < 0. \end{array}}$$

ex. DETERMINE STABILITY OF ALL FIXED POINTS OF $x_{n+1} = x_n^3 + \frac{3}{4} x_n$

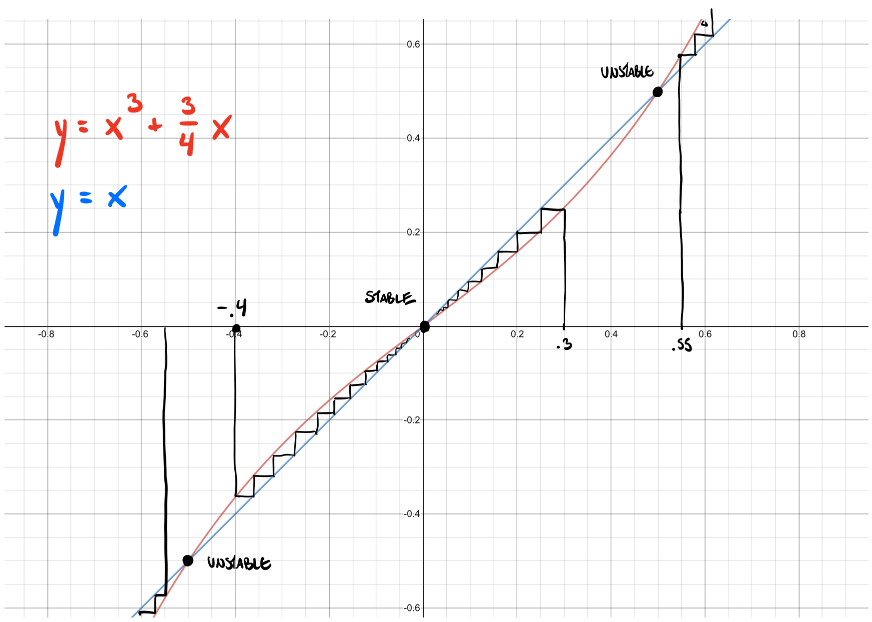
FIND FIXED POINTS: FIXED POINT EQUATION $p = f(p)$ (SOLVE)

$$p = p^3 + \frac{3}{4} p \quad \Rightarrow \quad 0 = p^3 - \frac{1}{4} p = p\left(p^2 - \frac{1}{4}\right) = p\left(p + \frac{1}{2}\right)\left(p - \frac{1}{2}\right)$$

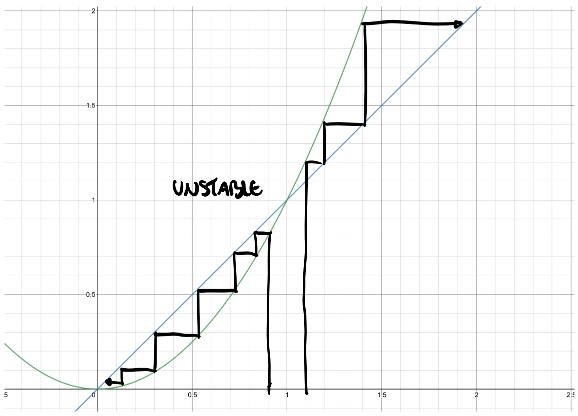
FIXED POINTS : $0, \pm \frac{1}{2}$.

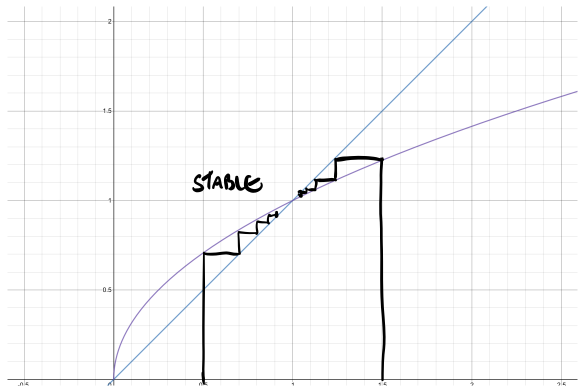
$$f(x) = x^3 + \frac{3}{4} x \quad \Rightarrow \quad f'(x) = 3x^2 + \frac{3}{4}$$

| FIXED POINTS $p$ | $f'(p)$ | CLASSIFY |
|---|---|---|
| $0$ | $3/4$ | $\lvert f'(p)\rvert < 1$, $f'(p) > 0$ $\Rightarrow$ STABLE |
| $1/2$ | $6/4$ | $\lvert f'(p)\rvert > 1$ $\Rightarrow$ UNSTABLE |
| $-1/2$ | $6/4$ | $\lvert f'(p)\rvert > 1$ $\Rightarrow$ UNSTABLE |



31. The point $p = 1$ is a fixed point of both $f(x) = x^2$ and its *inverse* $f^{-1}(x) = \sqrt{x}$ (for $x \geq 0$). (a) Use cobweb graphs to show that $p$ is stable under one of these functions and unstable under the other. (b) Confirm these observations using the derivative.




$y = f(x) = x^2$ $\quad y = x$

$f'(x) = 2x \Rightarrow f'(1) = 2$

$|f'(1)| > 1$

$p = 1$ : UNSTABLE

INVERSE FUNCTIONS

DERIV. AT FIXED POINTS ARE RECIPRICOLS.

$y = f(x) = \sqrt{x}$ $\quad y = x$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(1) = \frac{1}{2}$

$|f'(1)| < 1$

$p = 1$ : STABLE.

$$\boxed{\text{INVERSE FUNCTION THEOREM: } \quad (f^{-1})'(p) = \frac{1}{f'(f^{-1}(p))} \quad \text{FOR ALL } p}$$

ASSUMING $f$ IS 1-TO-1, $f(p) = p \iff f^{-1}(p) = p$.

$$(f^{-1})'(p) = \frac{1}{f'(p)}$$

$\therefore |f'(p)| < 1 \iff |(f^{-1})'(p)| > 1$

$\therefore$ $p$ IS STABLE FIXED POINT OF $f \iff p$ IS UNSTABLE FIXED POINT OF $f^{-1}$. ✓

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**ex.** DETERMINE THE STABILITY OF THE FIXED POINTS OF

(a) $f(x) = \dfrac{1}{e^x}$ $\qquad$ (b) $g(x) = \ln\left(\dfrac{1}{x}\right)$

(NOTE THAT IT IS DIFFICULT TO DETERMINE THE FIXED POINTS ALGEBRAICALLY! WE WILL LEARN A VERY GOOD WAY TO APPROXIMATE THE FIXED POINTS SOON!)

# Proof of Stability & Oscillation Thm.

If $|f'(p)| > 1$ then $p$ is unstable
If $|f'(p)| < 1$ then $p$ is stable
Solutions oscillate if & only if $f'(p) < 0$.

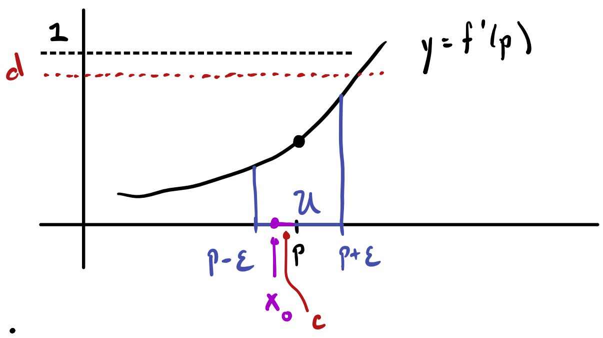
Stability : Two cases

(i) If $|f'(p)| < 1$.

Assuming $f'$ is continuous at $p$,

Then $\exists$ (there exists) open interval $\mathcal{U} = (p-\varepsilon, p+\varepsilon)$, $\varepsilon > 0$,

and $0 < d < 1$ such that $|f'(x)| \leq d \quad \forall x \in \mathcal{U}$.



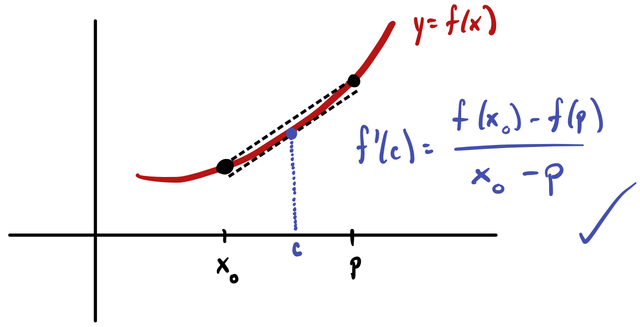

Let $x_0 \in \mathcal{U}$.

Then Mean Value Theorem $\Rightarrow$ $\exists$ $c$ between $p$ & $x_0$ such that

(*) $$f(x_0) - f(p) = f'(c)(x_0 - p)$$

$$x_1 - p = f'(c)(x_0 - p)$$

$$|x_1 - p| = |f'(c)|\,|x_0 - p|.$$




Since $c \in \mathcal{U}$, $|x_1 - p| \leq d\,|x_0 - p| < |x_0 - p|$, $d < 1$

$\therefore$ $x_1 \in \mathcal{U}$

Same argument, with $x_0$ replaced by $x_1$, shows

$$|x_2 - p| \leq d\,|x_1 - p| \leq d\left(d\,|x_0 - p|\right) = d^2\,|x_0 - p|$$

$$\vdots$$

$$|x_n - p| \leq d^n\,|x_0 - p|$$

$$\Rightarrow \quad 0 \leq \lim_{n\to\infty} |x_n - p| \leq \lim_{n\to\infty} d^n\,|x_0 - p| = |x_0 - p| \lim_{n\to\infty} d^n = 0$$

$\therefore$ $x_n$ converges to $p$. ✓

(i) IF $|f'(p)| > 1$.

THEN $\exists$ OPEN INTERVAL $\mathcal{U} = (p-\varepsilon, p+\varepsilon)$, $\varepsilon > 0$, AND $\exists\ d > 1$ SUCH THAT $|f'(x)| \geq d \quad \forall x \in \mathcal{U}$.

LET $x_0 \in \mathcal{U}$

THEN MVT $\Rightarrow$ $f(x_0) - f(p) = f'(c)(x_0 - p)$ FOR SOME $c$ BETWEEN $x_0$ & $p$

$$\underbrace{f(x_0)}_{\downarrow\ x_1} - \underbrace{f(p)}_{\downarrow\ p} = f'(c)(x_0-p)$$

$$x_1 - p = f'(c)(x_0 - p)$$

$c \in \mathcal{U} \Rightarrow f'(c) \geq d > 1$

$\Rightarrow |x_1 - p| \geq d\,|x_0 - p|$   $x_1$ IS FURTHER FROM $p$ THAN $x_0$ IS.

$\Rightarrow$ FIXED POINT $p$ IS NOT STABLE. $\Rightarrow$ UNSTABLE ✓

## OSCILLATION:

IF $f'(p) > 0$ THEN WE CHOOSE $\mathcal{U} = (p-\varepsilon, p+\varepsilon)$, $\varepsilon > 0$ s.t. $f'(x) > 0 \quad \forall x \in \mathcal{U}$

$\therefore$ IF $x_0 - p < 0$   $(x_0 < p)$

(*) THEN $f(x_0) - p = \underbrace{f'(c)}_{\text{POS}}\underbrace{(x_0 - p)}_{\text{NEG}} \Rightarrow f(x_0) - p < 0$

$$f(x_0) < p$$

$$x_1 < p.$$

SAME SIDE OF $p$ AS $x_0$. ✓

IF $x_0 - p > 0$   $(x_0 > p)$

THEN $f(x_0) - p = \underbrace{f'(c)}_{\text{POS}}\underbrace{(x_0 - p)}_{\text{POS}} \Rightarrow f(x_0) - p > 0$

$$f(x_0) > p$$

$$x_1 > p.$$

SAME SIDE OF $p$ AS $x_0$. ✓

IF $f'(p) < 0$ THEN WE CHOOSE $\mathcal{U} = (p-\varepsilon, p+\varepsilon)$, $\varepsilon > 0$
s.t. $f'(x) < 0 \ \forall x \in \mathcal{U}$

$\therefore$ IF $x_0 - p < 0$ $\quad (x_0 < p)$

THEN $f(x_0) - p = \underbrace{f'(c)}_{\text{NEG}} \underbrace{(x_0 - p)}_{\text{NEG}} \Rightarrow f(x_0) - p > 0$

$f(x_0) > p$

$x_1 > p$. OPP. SIDE OF $p$ AS $x_0$. ✓

IF $x_0 - p > 0$ $\quad (x_0 > p)$

THEN $f(x_0) - p = \underbrace{f'(c)}_{\text{NEG}} \underbrace{(x_0 - p)}_{\text{POS}} \Rightarrow f(x_0) - p < 0$

$f(x_0) < p$

$x_1 < p$. OPP. SIDE OF $p$ AS $x_0$. ✓