§ 3.2 AUTONOMOUS EQ'S & THEIR DYNAMICS.

DYNAMICS : $x_{n+1} = f(x_n)$ $\Rightarrow$ $x_1 = f(x_0)$
$x_2 = f \circ f(x_0)$
$x_3 = f \circ f \circ f(x_0)$
$\vdots$

$$P_{n+1} = rP_n\left(1 - \frac{P_n}{C}\right) \Rightarrow P_{n+1} = f(P_n), \quad f(x) = rx\left(1 - \frac{x}{C}\right)$$

$$P_{n+1} = rP_n e^{-P_n/N} \Rightarrow P_{n+1} = f(P_n), \quad f(x) = rxe^{-x/N}$$

WHAT SORT OF INSIGHTS DOES THE GRAPH $y = f(x)$ PROVIDE?

In Exercises 6–9 sketch the graph of y = f(x) *and find the coordinates of the maximum point.*

6. $f(x) = rx(1 - x^2 10^{-6})$ for: (a) $r = 1/2$; (b) $r = 2$.
7. $f(x) = \frac{rx^2}{3000}(1 - x/3000)$ for: (a) $r = 1$; (b) $r = 6$.
8. $f(x) = \frac{rx^2}{1000}(1 - x/1000)^2$ for: (a) $r = 1$; (b) $r = 10$.
9. $f(x) = \frac{rx^2}{500}e^{-x/500}$ for: (a) $r = 1$; (b) $r = 3$.

NOTE: $$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{IF } |a| < 1 \\ 1 & \text{IF } a = 1 \\ \text{D.N.E.} & \text{OTHERWISE} \end{cases}$$

PROOF: •) SUPPOSE $a > 1$.

THEN $$\lim_{n\to\infty} a^n = \lim_{n\to\infty} e^{n \ln a} = \infty$$

•) SUPPOSE $0 < a < 1$.

THEN $$\lim_{n\to\infty} a^n = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{a}\right)^n}, \quad \frac{1}{a} > 1$$
$$= 0$$

·) IF $a < 0$, THEN $\lim_{n\to\infty} |a^n| = \lim_{n\to\infty} |a|^n = \begin{cases} 0 & \text{IF } |a| < 1 \\ \infty & \text{IF } |a| > 1 \end{cases}$

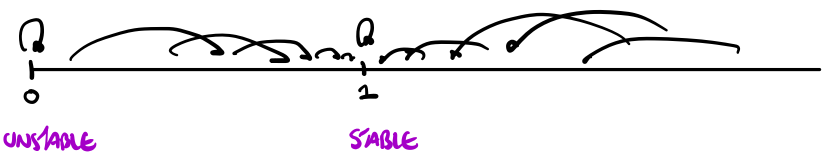
·) IF $a = 0$, $\lim_{n\to\infty} a^n = 0$.

EXACT SOLUTIONS FOR AUTONOMOUS NON-LINEAR EQ'S:

$$x_{n+1} = x_n^{a} \quad \Rightarrow \quad x_n = x_0^{a^n} \qquad \text{(EXPLAIN)}$$

INVERSE FUNCTIONS

DISCUSS ASYMPTOTIC BEHAVIOR OF x_n FOR $x_{n+1} = x_n^{1/2}$, $x_0 \geq 0$



DISCUSS ASYMPTOTIC BEHAVIOR OF x_n FOR $x_{n+1} = x_n^{2}$

DEFINITION

A fixed point p of $x_{n+1} = f(x_n)$ is **locally stable** or an **attractor** if $|x_{n+1} - p| < |x_n - p|$, and so $\lim_{n\to\infty} x_n = p$, for all x_0 close to p. Otherwise, p is **unstable** or a **repeller.**

As the word *locally* implies, stability of a fixed point p of a nonlinear function generally involves an interval around p in which solutions converge to p. The largest such interval is sometimes called the **basin of attraction** of the fixed point. What happens outside that interval is irrelevant to local stability. Note that this issue does not arise for linear equations, since either all solutions converge to the fixed point p, or all diverge to $\pm\infty$ (except for p itself). That type of stability is often called **global stability,** and the corresponding basin of attraction consists of the entire real line. Since local stability is the only type of stability we consider for nonlinear equations, we often dispense with the words *local* and *locally*.

In Exercises 25–28 find all fixed points and determine their stability using the formula for the exact solution $x_n = x_0^{a^n}$. Find the basins of attraction for those that are stable.

25. $x_{n+1} = x_n^2$

26. $x_{n+1} = x_n^{1/3}$

27. $x_{n+1} = 1/x_n^2$

28. $x_{n+1} = 1/\sqrt{x_n}$

27. FIXED POINT EQ: $p = \frac{1}{p^2} \Rightarrow p^3 = 1 \Rightarrow p = 1.$

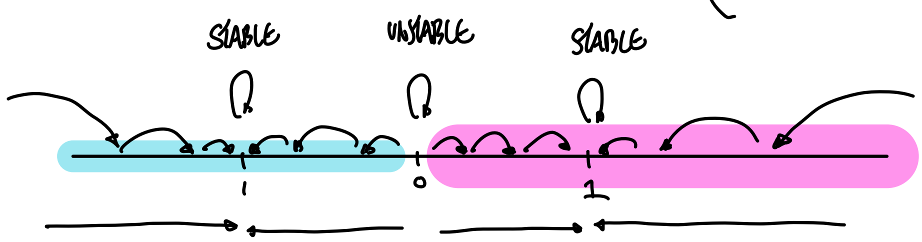
$x_n = x_0^{(-2)^n}$ EVEN ITERATES: $\lim\limits_{n\to\infty} x_0^{(-2)^{2n}} = \lim\limits_{n\to\infty} x_0^{4^n} = \begin{cases} 0 & \text{IF } |x_0| < 1 \\ 1 & \text{IF } x_0 = 1 \\ \text{DNE} & \text{OTHERWISE} \end{cases}$

$\therefore$ $p = 1$ IS NOT STABLE

26. FIXED POINT EQ: $p = p^{1/3} \Rightarrow p^3 = p \Rightarrow p(p^2 - 1) = 0$

3 FIXED POINTS: $p = -1, 0, 1$

$x_n = x_0^{(1/3)^n} \Rightarrow \lim\limits_{n\to\infty} x_0^{(1/3)^n} = \begin{cases} 0 & \text{IF } x_0 = 0 \\ 1 & \text{IF } x_0 > 0 \\ -1 & \text{IF } x_0 < 0 \end{cases}$



0 IS UNSTABLE FIXED POINT

-1 IS STABLE FIXED POINT WITH BASIN OF ATTRACTION $(-\infty, 0)$

1 IS STABLE FIXED POINT WITH BASIN OF ATTRACTION $(0, \infty)$

10. For the equation $x_{n+1} = bx_n^a$ where a and b are positive constants:

(a) Compute $x_1, \ldots, x_5$, in terms of x_0.

(b) Can you construct a formula for the exact solution x_n for all n?

(c) Use the geometric series to simplify your solution in part (b).

(a) $x_1 = bx_0^a$

$x_2 = bx_1^a = b(bx_0^a)^a = b(b^a x_0^{a^2}) = b^{1+a} x_0^{a^2}$

$x_3 = bx_2^a = b(b^{1+a} x_0^{a^2})^a = b(b^{a+a^2} x_0^{a^3})$

$= b^{1+a+a^2} x_0^{a^3}$

$\vdots$

(b) $x_n = b^{1+a^2+\ldots a^{n-1}} x_0^{a^n}$

(c) $1 + a + a^2 + \ldots + a^{n-1} = \sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a} = \frac{a^n - 1}{a-1}$

$x_n = b^{\frac{1-a^n}{1-a}} x_0^{a^n}$

In Exercises 25–28 find all fixed points and determine their stability using the formula for the exact solution $x_n = x_0^{a^n}$. *Find the basins of attraction for those that are stable.*

25. $x_{n+1} = x_n^2$

26. $x_{n+1} = x_n^{1/3}$

27. $x_{n+1} = 1/x_n^2$

28. $x_{n+1} = 1/\sqrt{x_n}$

In Exercises 29–32 find all fixed points and determine their stability using the formula for the exact solution (see Exercise 10). Find the basins of attraction for those that are stable.

29. $x_{n+1} = x_n^2/2$

30. $x_{n+1} = 4x_n^3$

31. $x_{n+1} = \sqrt{x_n}/3$

32. $x_{n+1} = 5\sqrt{x_n}$

29. $x_{n+1} = \frac{1}{2} x_n^2$

FIXED POINT(S): $p = \frac{1}{2} p^2$

$0 = p^2 - 2p = p(p-2) \implies p = 0, 2.$

EXACT SOLUTION: $x_n = \left(\frac{1}{2}\right)^{2^n - 1} x_0^{2^n} =$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2\left(\frac{x_0}{2}\right)^{2^n} = \begin{cases} 0 & \text{IF } \left|\frac{x_0}{2}\right| < 1 \iff -2 < x_0 < 2 \\ 1 & \text{IF } \frac{x_0}{2} = 1 \iff x_0 = 2 \\ \text{D.N.E.} & \text{OTHERWISE} \end{cases}$$

$\therefore$ $p = 0$ IS STABLE WITH BASIN OF ATTRACTION $(-2, 2)$

$p = 2$ IS UNSTABLE.

32. $x_{n+1} = 5\, x_n^{\frac{1}{2}}$

FIXED PNTS: $p = 5\sqrt{p} \implies p^2 - 25p = 0 \implies p(25 - p) = 0$

$p = 0, 25$

EXACT SOL'N: $x_n = 5^{\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}} x_0^{\left(\frac{1}{2}\right)^n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 5^{\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}} \cdot \lim_{n\to\infty} x_0^{\left(\frac{1}{2}\right)^n}$$

$$= 25 \cdot \lim_{n\to\infty} x_0^{\left(\frac{1}{2}\right)^n} = \begin{cases} 0 & \text{IF } x_0 = 0 \\ 1 & \text{IF } x_0 > 0 \end{cases}$$

$\therefore$ $p = 0$ UNSTABLE, $p = 25$ STABLE WITH BASIN OF ATTRACTION $(0, \infty)$