Chapter 2 : Linear Models $x_{n+1} = a_n x_n + b_n$

Special Cases:

Sums : $S_n = \sum_{k=0}^{n-1} b_k \Rightarrow S_{n+1} = S_n + b_n$

Products : $P_n = \prod_{k=0}^{n-1} a_k \Rightarrow P_{n+1} = a_n P_n$ (homogeneous)

Autonomous : $x_{n+1} = a x_n + b$, a & b constants.

$x_{n+1} = f(x_n)$, $f(x) = ax + b$

All differentiable functions look like linear functions when zoomed in.

Polynomials? rational? algebraic? exponential?

Chapter 3 : Non-linear Models $x_{n+1} = f(x_n)$

(e.g. $x_{n+1} = 3x - 3x^2$)

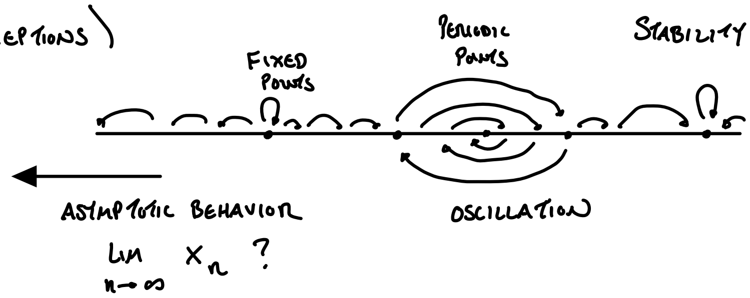
$x_1 = 3x_0 + 3x_0^2$

$x_2 = 3(3x_0 + 3x_0^2) + 3(3x_0 + 3x_0^2)^2 = 9x_0 + 36x_0^2 + 54x_0^3 + 27x_0^4$

$\vdots$

No exact solutions! (with few exceptions)

We can still study dynamics

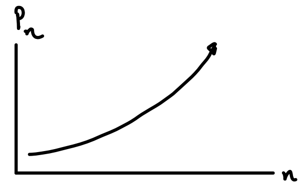


§3.1 Some Non-Linear Models

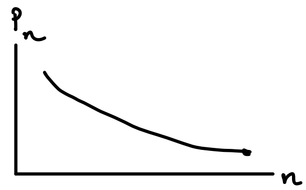
- Developing & refining models
  - Two new population models
  - Two infectious diseas model

Population Models: Linear $P_{n+1} = r P_n$, $r \geq 0$ growth rate

$\Rightarrow P_n = r^n P_0$



$r > 1$



$0 < r < 1$

Growth rate r should depend on population P_n

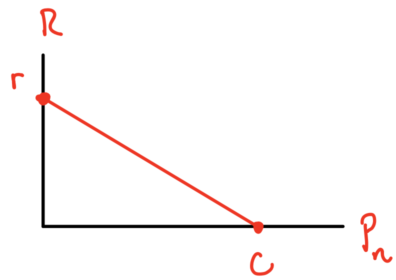
Principle of **density dependence**

Replace r with $R(P_n)$.

Then $P_{n+1} = r P_n \rightsquigarrow P_{n+1} = \underbrace{R(P_n)}\, P_n$

Should decrease as P_n gets bigger.

Choice #1: $R(P_n) = r\left(1 - \frac{P_n}{C}\right)$



Def: Carrying capacity is the max population the environment can sustain.

If P_n reaches this value then $R(P_n) = 0$ (extinction).

Non Linear Pop Model I (Logistic Model)

$$P_{n+1} = R(P_n) P_n, \quad R(P_n) = r\left(1 - \frac{P_n}{C}\right)$$

$$P_{n+1} = r P_n \left(1 - \frac{P_n}{C}\right), \quad 0 \leq r \leq 4, \quad 0 \leq P_0 \leq C$$

If P_n reaches C then $P_{n+1} = 0$ (extinction).

DEMO

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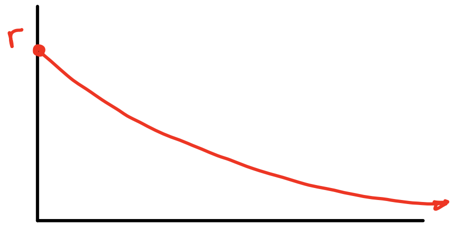
ex. $x_{n+1} = 1.2\, x_n \left(1 - \frac{x_n}{400{,}000}\right)$

Then change to 2.5, 3, 3.5, 3.8 !

Choice #2: $R(P_n) = r e^{-P_n/N}$, $N > 0$



$R(P_n) > 0$ (No extinction)

Nonlinear Pop Model II

$$P_{n+1} = R(P_n) P_n \quad , \quad R(P_n) = r e^{-P_n/N}$$

$$P_{n+1} = r P_n e^{-P_n/N} \quad , \quad 0 \le r \quad , \quad N > 0$$

Let $f(x) = r x e^{-x/N}$ so $P_{n+1} = f(P_n)$

$$f'(x) = r e^{-x/N} - \frac{rx}{N} e^{-x/N} = 0$$

$$r\left(1 - \frac{x}{N}\right) = 0$$

$$x = N \quad \underline{\underline{\text{MAX}}} \quad (f' : \oplus \to \ominus)$$

$\therefore$ N is the population that maximizes the pop of next generation

DEMO

Set very high P_0

Infectious Disease Models

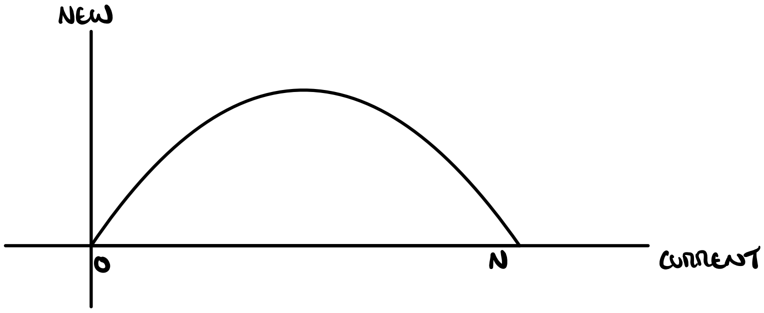
Let I_n : # infected individuals at time n.

Start simple: $I_{n+1} = I_n - r I_n$ + new cases

r = recovery rate , $0 \le r \le 1$

Generally Accepted Principle: New Cases is proportional to the product (# infected)(# susceptible)

Method 1: New Cases $= k I_n (N - I_n)$



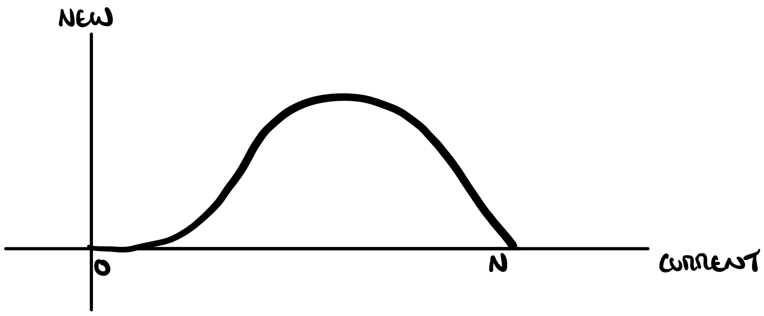
Nonlinear Infection Model I

$$I_{n+1} = (1-r)I_n + k I_n (N - I_n) = (1-r)I_n + \underbrace{kN}_{s} I_n \left(1 - \frac{I_n}{N}\right)$$

$0 \le r \le 1$, $N > 0$ $\quad$ $0 \le 1-r+s \le 4$

$0 \le I_0 \le N$

Method 2 New Cases $= k I_n^2 (N - I_n)$



Nonlinear Infection Model II

$$I_{n+1} = (1-r)I_n + k I_n^2 (N - I_n) = (1-r)I_n + \underbrace{kN}_{s/N} I_n^2 \left(1 - \frac{I_n}{N}\right)$$

$0 \le r \le 1$, $N > 0$ $\quad$ $0 \le 1-r+s \le 4$

$0 \le I_0 \le N$