§2.6 Dynamics of Linear Eq's

(Recall) Given a function f such that $\text{Ran}(f) \subseteq \text{Dom}(f)$

A single value x_0 can produce an infinite sequence

$x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_{n+1} = f(x_n)$.

Each choice of x_0 produces its own sequence, called a solution to the equation/model/system $x_{n+1} = f(x_n)$.

The **dynamics** of a system refers to any set of properties/characteristics that some/all solutions have.

1. Does $\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} f^n(x_0)$ exist?

   If yes, solution converges

   If no, solution diverges

   Maybe it depends on x_0, maybe not.

2. Are solutions monotone?

   $x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$

   $x_0 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq x_{n+1} \geq \dots$

   Maybe it depends on x_0, maybe not.

Definition:

Non decreasing or Non increasing

e.g. Fibonacci Seq.

e.g. $F(x) := \max\left(\{ f(a) : 0 \leq a \leq x \} \right)$ "Greatest so far"

CONVERGENCE / DIVERGENCE:

LINEAR EQUATION $x_{n+1} = f(x_n) = a x_n + b$

EXACT SOLUTION $x_n = a^n x_0 + \frac{1-a^n}{1-a} \cdot b \quad , \quad a \neq 1$

$$= \frac{b}{1-a} + a^n \left(x_0 - \frac{b}{1-a} \right)$$

<u>NOTE:</u> IF $x_0 = \frac{b}{1-a}$ THEN $x_1 = \frac{b}{1-a}$ ALSO

AND SO $x_n = \frac{b}{1-a}$ FOR ALL n.

SOLUTION IS CONSTANT.

<u>DEF:</u> p IS A FIXED POINT OF THE EQUATION $x_{n+1} = f(x_n)$
IF $\underbrace{p = f(p)}_{\text{FIXED POINT EQUATION}}$.

<u>THM:</u> $p = \frac{b}{1-a}$ IS A FIXED POINT OF THE EQ $x_{n+1} = a x_n + b$.

$$\lim_{n\to\infty} x_n = \underbrace{\frac{b}{1-a}}_{\text{CONSTANT}} + \lim_{n\to\infty} a^n \underbrace{\left(x_0 - \frac{b}{1-a} \right)}_{\text{CONSTANT}}$$

(RECALL) $\lim_{n\to\infty} |a^n| = \begin{cases} 0 & \text{IF } |a| < 1 \\ 1 & \text{IF } |a| = 1 \\ \infty & \text{IF } |a| > 1 \end{cases}$

THM: Let $p = \frac{b}{1-a}$ be the fixed point of $x_{n+1} = ax_n + b$.

- If $|a| < 1$ then $\lim_{n\to\infty} x_n = p$

  Solutions converge to p for all x_0

  p is a stable fixed point. (attractor)

- If $|a| > 1$ then $\lim_{n\to\infty} |x_n| = \infty$

  Solutions diverge for all $x_0 \neq p$

  p is an unstable fixed point (repeller)

Monotonicity:

$$x_n = \frac{b}{1-a} + a^n\left(x_0 - \frac{b}{1-a}\right)$$

$$x_n = p + a^n(x_0 - p)$$ Assume $x_0 \neq p$.

THM: For $x_0 \neq p$

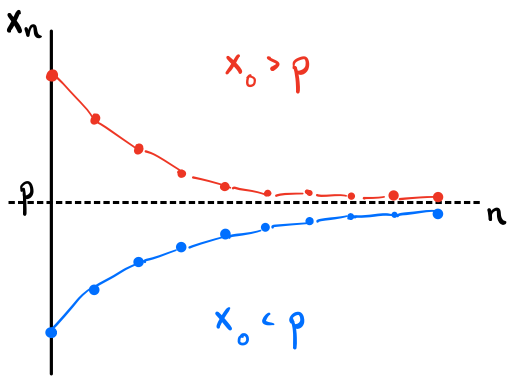
- If $a > 0$ then all x_n lie on same side of p

- If $a < 0$ then solution oscillates between values above/below p.

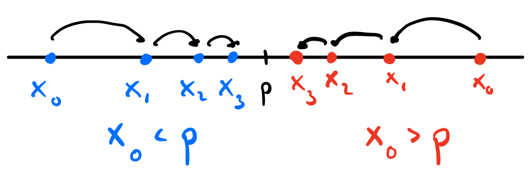
Summary:

	Converge p is stable fixed point	**Diverge** p is unstable fixed point
Solution is **Monotonic**	$0 < a < 1$	$1 < a$
Solution **Oscillates**	$-1 < a < 0$	$a < -1$

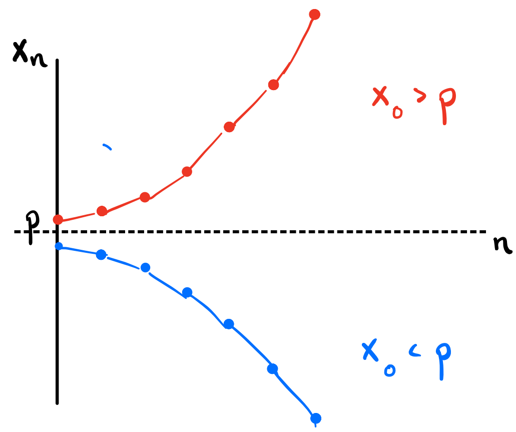
Solution is Monotonic — Converge ($0 < a < 1$)



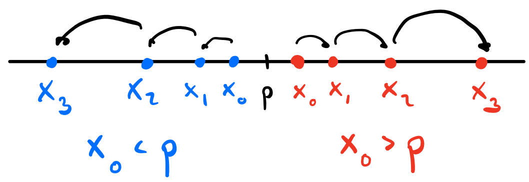
$0 < a < 1$



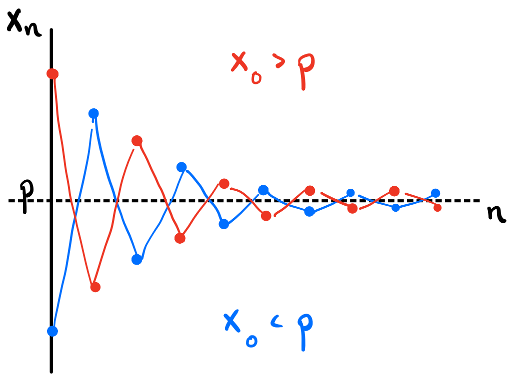
Solution is Monotonic — Diverge ($1 < a$)



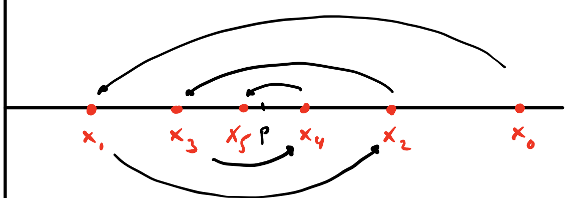
$1 < a$



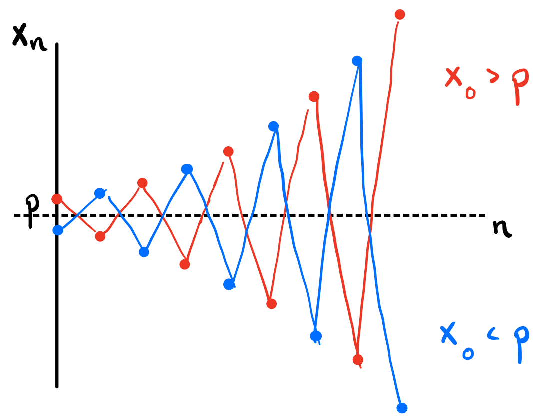
Solution Oscillates — Converge ($-1 < a < 0$)



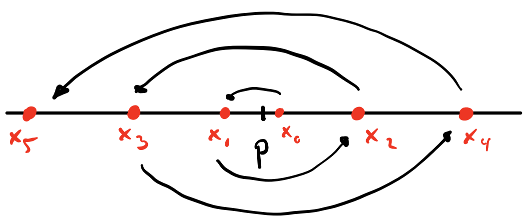
$-1 < a < 0$



Solution Oscillates — Diverge ($a < -1$)



$a < -1$



In Exercises 1–8 find the fixed point and determine both its stability and whether or not oscillation occurs.

1. $x_{n+1} = 2x_n - 3/4$
2. $x_{n+1} = 0.7x_n$
3. $x_{n+1} = \frac{1}{2}x_n + 4$
4. $x_{n+1} = (x_n + 1)/2$
5. $x_{n+1} = -1.75x_n$
6. $x_{n+1} = 0.9(1 - x_n)$
7. $x_{n+1} = 4(x_n - 1)/5$
8. $x_{n+1} = (7 - 3x_n)/2$

In Exercises 9–14 find the equilibria of the models described in previous exercises. Also determine both stability and whether or not oscillation occurs.

9. Exercise 23 of Section 2.3
10. Exercise 24 of Section 2.3
11. Exercise 25 of Section 2.3
12. Exercise 21 of Section 2.4
13. Exercise 22 of Section 2.4
14. Exercise 23 of Section 2.4
15. Suppose that there are currently 25,000 unemployed workers in some state. Each month 8% of all those unemployed find jobs but another 1500 become unemployed.
    (a) How many will be unemployed 6 months from now?
    (b) At what level will the number of unemployed workers stabilize over time?

17. Suppose that someone always carries an unpaid balance on a certain credit card. Each month the credit card company charges 1% interest on any previous unpaid balance, and the person pays off 10% of that previous balance. Also, during each month another $200 is charged on that credit card.
    (a) Write an iterative equation for the monthly unpaid balance U_n.
    (b) What level will the unpaid balance gradually approach?
18. Suppose that each year 2% of all the trees in a certain forest are destroyed naturally. Also each year 5000 mature trees are harvested for lumber, but 7500 new trees are either planted or sprout up on their own.
    (a) Write an iterative equation for the yearly number of trees T_n in that forest.
    (b) If there are currently estimated to be 100,000 trees in that forest, what is the maximum number of trees the forest will ever have?

5. (a) Consider the linear model

$$x_{n+1} = 1.15x_n - 360.$$

   i. (2 points) For what values of x_0 (if any) do solutions monotonically increase?
   ii. (2 points) For what values of x_0 (if any) do solutions monotonically decrease?
   iii. (2 points) For what values of x_0 (if any) do solutions oscillate?

   (b) Consider the model

$$x_{n+1} = .75x_n + 480.$$

   i. (2 points) For what values of x_0 (if any) do solutions monotonically increase?
   ii. (2 points) For what values of x_0 (if any) do solutions monotonically decrease?
   iii. (2 points) For what values of x_0 (if any) do solutions oscillate?

   (c) Consider the equation

$$x_{n+1} = -2.2x_n + 640.$$

   i. (2 points) For what values of x_0 (if any) do solutions monotonically increase?
   ii. (2 points) For what values of x_0 (if any) do solutions monotonically decrease?
   iii. (2 points) For what values of x_0 (if any) do solutions oscillate?