LAST TIME:

EXACT SOLUTION TO AUTONOMOUS LINEAR EQUATION

GIVEN x_0 & $x_{n+1} = ax_n + b$, THE EXACT SOLUTION IS

$$x_n = a^n x_0 + b\left(\frac{1-a^n}{1-a}\right) \qquad (*)$$

ALTERNATIVE DERIVATION OF $(*)$:

"CHANGE OF VARIABLES"

$x_{n+1} = ax_n + b$. LET $u_n = x_n - \frac{b}{1-a}$, $u_{n+1} = x_{n+1} - \frac{b}{1-a}$.

$$\rightsquigarrow \quad u_{n+1} + \frac{b}{1-a} = a\left(u_n + \frac{b}{1-a}\right) + b = au_n + \frac{ab}{1-a} + \frac{b(1-a)}{1-a}$$

$$= au_n + \frac{b}{1-a}$$

$\Rightarrow$ $u_{n+1} = au_n$ (HOMOGENEOUS!)

$\therefore$ $u_n = a^n u_0$ $\quad \rightsquigarrow \quad x_n - \frac{b}{1-a} = a^n\left(x_0 - \frac{b}{1-a}\right) = a^n x_0 - \frac{ba^n}{1-a}$

$$x_n = a^n x_0 - \frac{ba^n}{1-a} + \frac{b}{1-a}$$

$$x_n = a^n x_0 + b\left(\frac{1-a^n}{1-a}\right) \quad ✓$$

$$\begin{tikzcd}
u_0 \arrow[r, bend left, "\times a"] & u_1 \arrow[r, bend left, "\times a"] & u_2 \arrow[r, bend left, "\times a"] & \cdots \arrow[r, bend left, "\times a"] & u_n \arrow[d, "x_n = u_n + \frac{b}{1-a}"] \\
x_0 \arrow[u, "u_0 = x_0 - \frac{b}{1-a}"] \arrow[r, bend left] & x_1 = ax_0 + b \arrow[r, bend left] & x_2 = ax_1 + b \arrow[r, bend left] & \cdots \arrow[r, bend left] & x_n = ax_{n-1} + b
\end{tikzcd}$$

$\boxed{u_{n+1} = au_n}$ HOMOGENEOUS

$\boxed{x_{n+1} = ax_n + b}$ NON-HOMOGENEOUS

Example 4. Suppose you invest \$1000 into an account that earns 6% annual interest, compounded monthly. Additionally, every month thereafter you invest an additional \$200. If this continues for 30 years, how much is the investment worth? How much of this is interest?

Example 5. Suppose you take out a 5 year loan of \$10,000 to pay for a car. The loan charges 4.8% annual interest compounded monthly. How much do you owe the lender each month? How much will end up paying back the lender?

3. Two days ago a stock debuted on the New York Stock Exchange and its closing price (in dollars) was $x_0 = 48$. Yesterday its closing price was $x_1 = 18$, and today its closing price was $x_2 = 33$.
   (a) (8 points) Find an autonomous linear model for x_{n+1} in terms of x_n.
   (b) (2 points) Use your model to predict the closing price of the stock tomorrow.

(c) FIND $\lim_{n\to\infty} x_n$ & INTERPRET THIS VALUE.

(*) SHOULD YOU BUY THIS STOCK? WHEN?

2 State Markov Process/Chain

Def: A Markov Process/Chain is a probabilistic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

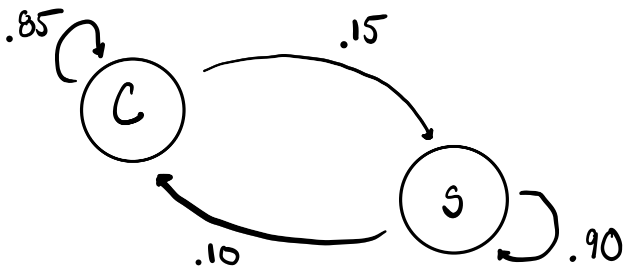
ex. Suppose an area with constant population of 100,000, and each year

- 15% of city dwellers move to the suburbs
- 10% of suburbanites move to the city.

Let C_n & S_n be city & suburbs population after n years,

$C_0 = S_0 = 50{,}000$.

Transition Diagram:



Equations:
$$C_{n+1} = .85\,C_n + .10\,S_n$$
$$S_{n+1} = .15\,C_n + .90\,S_n$$
$$C_n + S_n = 100{,}000$$

$$C_{n+1} = .85\,C_n + .10(100{,}000 - C_n)$$
$$C_{n+1} = .75\,C_n + 10{,}000$$

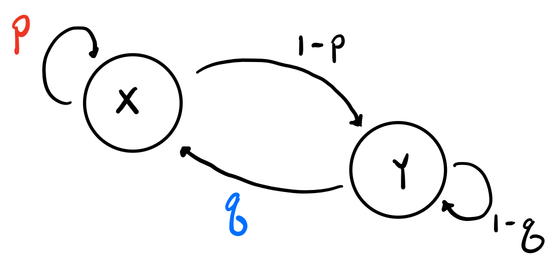
Exact Solution:

$$\boxed{x_n = a^n x_0 + \left(\frac{1-a^n}{1-a}\right) b} \implies C_n = .75^n(50{,}000) + \left(\frac{1-.75^n}{1-.75}\right)10{,}000$$

and $\lim_{n\to\infty} C_n = \dfrac{10{,}000}{1-.75} = \underline{\underline{40{,}000}}$

More Generally, X_n = proportion of fixed population in state X

Y_n = proportion of fixed population in state Y



$$X_{n+1} = pX_{n-1} + qY_n = pX_{n-1} + q(1-X_n)$$
$$X_{n+1} = (p-q)X_n + q \implies X_n = (p-q)^n X_0 + \left(\frac{1-(p-q)^n}{1-(p-q)}\right) q$$

4. The box office at Yankee Stadium discovered that 30% of season ticket holders who attand (A) one home game do not attend (N) the next home game, and 50% of season ticket holder who do not attend (N) one home game do attend (A) the next home game.

   (a) (4 points) Draw and label a transition diagram that summarizes the box office's discovery.

   (b) (6 points) Construct an iterative equation the models how the proportion of season ticket holders that attend games changes from one game to the next.

   (c) (6 points) Suppose 92% of season ticket holders attend the first home game of the season. By the end of the season, what proportion of season ticket holders should the box office expect to attend home games?

2. Suppose your credit card balance is \$1200 and the credit card company charges 24% annual interest (APR) compounded monthly. What size monthly payment must you make to pay off your debt in 6 months?

3. Suppose you can afford a monthly mortgage payment of \$1500 per month. If a 30 year mortgage charges an annual interest rate of 6% compounded monthly, how much money can you afford to borrow?

4. Suppose that of 1000 businesses being followed, 90% of those that make a profit in one year will also make a profit the following year and 10% will not, while 50% of those that didn't make profit in one year will make a profit the following year and 50% will not. If presently 750 of those businesses are making a profit and 250 are not, how many will and won't be making a profit in 5 years? In the long run?