LINEAR MODEL $x_{n+1} = a_n x_n + b_n$, RULE FOR a_n & b_n

& INITIAL VALUE x_0 GIVEN

SPECIAL CASES

DEF: HOMOGENEOUS

SETTING $a_n = 1$ $\forall n$ & $S_0 = 0$ | SETTING $b_n = 0$ $\forall n$, $P_0 = 1$

$S_{n+1} = S_n + b_n$ ← ITERATIVE EQUATIONS → $P_{n+1} = a_n P_n$

PREFERABLE IN COMP. SCI. (REQUIRES LOW MEMORY) ONLY NEED TO STORE S_n TO COMPUTE S_{n+1}.

$S_n = \sum_{k=0}^{n-1} b_k$ ← EXACT SOLUTIONS → $P_n = \prod_{k=0}^{n-1} a_k$

NEED TO STORE ALL n VALUES TO ADD UP TO COMPUTE S_n

INFINITE SUM, i.e. SERIES

$$\sum_{k=0}^{\infty} b_k = \lim_{n\to\infty} S_n$$

INFINITE PRODUCT

$$\prod_{k=0}^{\infty} a_k = \lim_{n\to\infty} P_n$$

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ex3. CONSIDER $S_{n+1} = S_n + r^n$, $S_0 = 0$ $(r \neq 1)$

FIND EXACT SOL'N & SIMPLIFY.

$$S_n = 1 + r + r^2 + \ldots + r^{n-1} = \sum_{k=0}^{n-1} r^k$$

$$-(rS_n = r + r^2 + \ldots + r^{n-1} + r^n)$$

$$(1-r)S_n = 1 - r^n$$

FINITE GEOMETRIC SERIES

$$\boxed{\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, \quad r \neq 1.}$$

ex4 GIVEN ANY x_0, LET

$x_{n+1} = r x_n + r x_0$

FIND EXACT SOL'N & SIMPLIFY.

$$x_n = x_0 \sum_{k=0}^{n} r^k = x_0 \left(\frac{1-r^{n+1}}{1-r}\right)$$

ex1 FIND EXACT SOLUTION TO

$$P_{n+1} = \left(1 + \frac{1}{n+1}\right) P_n, \quad P_0 = 1.$$

& SIMPLIFY.

$$P_n = \prod_{k=0}^{n-1} \left(1 + \frac{1}{n+1}\right)$$

$$= \cancel{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdot \ldots \cdot \frac{n+1}{\cancel{n}} = n+1$$

ex 2 CONSIDER $x_{n+1} = r x_n$, $x_0 = 1$.

FOR WHAT VALUES OF r DOES

$\lim_{n\to\infty} x_n$ EXIST?

$$= \lim_{n\to\infty} r^n = \begin{cases} 0 & \text{IF } |r| < 1 \\ 1 & \text{IF } r = 1 \\ \text{DNE} & \text{ELSE} \end{cases}$$

ex 5 WHAT IS THE SUM OF THE INFINITE GEOMETRIC SERIES

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \dots \quad ?$$

$$= \frac{1}{1-r} \quad \text{IF } |r| < 0 \text{ , DNE ELSE.}$$

ex 6. WHAT IS $4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \frac{64}{81} - \dots$? $= \frac{4}{1-(-\frac{2}{3})} = \frac{12}{5} = 2.4$

EXACT SOLUTION TO AUTONOMOUS LINEAR EQUATION

GIVEN x_0 & $x_{n+1} = a x_n + b$, THE EXACT SOLUTION IS

$$x_n = a^n x_0 + b\left(\frac{1-a^n}{1-a}\right) \quad (*)$$

$$x_1 = a x_0 + b$$
$$x_2 = a x_1 + b = a(a x_0 + b) + b = a^2 x_0 + ab + b$$
$$x_3 = a x_2 + b = a(a^2 x_0 + ab + b) + b = a^3 x_0 + a^2 b + ab + b$$
$$\vdots$$

16. Suppose that each month \$200 is deposited into a savings account with an original balance of \$1500, that pays 4% annual interest compounded monthly. (a) What will the account balance be 2 years later? (b) How much interest will the account earn during that time?

17. Suppose someone with a 401(k) plan presently worth \$50,000 wishes to retire 10 years from now. If the account is estimated to earn 8% annual interest compounded quarterly: (a) How much will be in the account at retirement time if each quarterly deposit is \$2000? (b) How much should each quarterly deposit be to have \$250,000 in the account at retirement time?

Alternative Derivation of (*):

CHANGE OF VARIABLES

$x_{n+1} = a x_n + b$. Let $u_n = x_n - \frac{b}{1-a}$, $u_{n+1} = x_{n+1} - \frac{b}{1-a}$.

$$\rightsquigarrow \quad u_{n+1} + \frac{b}{1-a} = a\left(u_n + \frac{b}{1-a}\right) + b = a u_n + \frac{ab}{1-a} + \frac{b(1-a)}{1-a}$$

$$= a u_n + \frac{b}{1-a}$$

$\Rightarrow u_{n+1} = a u_n$ (HOMOGENEOUS!)

$\therefore u_n = a^n u_0 \quad \rightsquigarrow \quad x_n - \frac{b}{1-a} = a^n\left(x_0 - \frac{b}{1-a}\right) = a^n x_0 - \frac{ba^n}{1-a}$

$$x_n = a^n x_0 - \frac{ba^n}{1-a} + \frac{b}{1-a}$$

$$x_n = a^n x_0 + b\left(\frac{1-a^n}{1-a}\right) \quad \checkmark$$