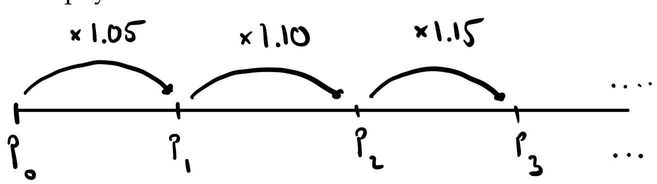
So far, we've looked at linear models of the form

$$x_{n+1} = ax_n + b$$

## More general linear equations

**Example 5.** Suppose you borrow $1,000 from a loan shark that charges compound interest every month. The structure of the loan is as follows: during the first month, the interest on the loan is 5%; during the second month, the interest on the loan (plus the previous month's interest) is 10%; during the third month, the interest on the loan (plus the previous months' interest) is 15%, etc.

1. Give a model for the amount owed at the end of each month if you never make any payments.




$P_1 = 1.05\,P_0$

$P_2 = 1.10\,P_1$

$P_3 = 1.15\,P_2$

$$\boxed{P_{n+1} = (1.05 + .05n)P_n}$$

2. Give a model in the case that you payback $200 at the end of the first month, $400 at the end of the second month, $600 at the end of the third month, etc.

$n=0$ $\quad P_1 = 1.05\,P_0 - 200$

$n=1$ $\quad P_2 = 1.10\,P_1 - 400$

$n=2$ $\quad P_3 = 1.15\,P_2 - 600$

$$\boxed{P_{n+1} = \underbrace{(1.05 + .05n)}_{a_n}P_n - \underbrace{200\,(n+1)}_{b_n}}$$

**Definition 2.** A linear model is an iterative equation of the form

$$x_{n+1} = a_n x_n + b_n$$

where $a_n$ and $b_n$ are functions of $n$.

**Definition 3.** An *autonomous* linear model is a linear model such that $a_n = a$ and $b_n = b$ are constant, i.e.

$$x_n = ax_n + b.$$

**Example 6.** Identify $a_n$ and $b_n$ is each of the following models.

- $$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 \\ x_2 &= 2 \cdot 1 \\ x_3 &= 3 \cdot 2 \cdot 1 \\ x_4 &= 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

  $\times 1$ $\times 2$ $\times 3$ $\times 4$

  $$x_{n+1} = a_n x_n + b_n \qquad \boxed{x_{n+1} = (n+1)x_n}$$

  $$a_n = n+1, \quad b_n = 0$$

- $$\begin{aligned} & x_0 = 1 \\ n=0 \quad & x_1 = 1 - 1/2 \\ n=1 \quad & x_2 = 1 - 1/2 + 1/3 \\ n=2 \quad & x_3 = 1 - 1/2 + 1/3 - 1/4 \\ n=3 \quad & x_4 = 1 - 1/2 + 1/3 - 1/4 + 1/5 \end{aligned}$$

  $$x_{0+1} = x_0 - \frac{1}{2}$$

  $$x_{1+1} = x_1 + \frac{1}{3}$$

  $$x_{2+1} = x_2 - \frac{1}{4}$$

  $$x_{n+1} = a_n x_n + b_n \qquad x_{n+1} = x_n + \frac{(-1)^{n+1}}{n+2}$$

  $$a_n = 1, \quad b_n = (-1)^{n+1} \cdot \frac{1}{n+2}$$

- $$x_{n+1} = \frac{4nx_n - 1}{2^n + 1}$$

  $$x_{n+1} = a_n x_n + b_n \qquad x_{n+1} = \underbrace{\frac{4n}{2^n+1}}_{a_n} x_n - \underbrace{\frac{1}{2^n+1}}_{b_n}$$

## Computing sums iteratively

LINEAR MODEL.

$$\sum_{i=1}^{N} c_i = \underbrace{\underbrace{\underbrace{c_1}_{S_1} + c_2}_{S_2} + c_3}_{S_3} + \ldots c_N$$

$S_n$

$n^{TH}$ PARTIAL SUM IS SUM OF FIRST $n$ TERMS

$$S_{n+1} = S_n + c_{n+1}$$

CALC II: $\lim_{n\to\infty} S_n = ?$

SOLUTION TO A MODEL IS A SEQUENCE

$S_1, S_2, S_3, \ldots$

THAT SATISFIES THE MODEL FOR ALL $n$.

The sequence of partial sums $S_n$, $0 \leq n \leq N$ ($S_0 = 0$) is a solution to the (non-autonomous) linear equation

$$S_{n+1} = S_n + c_{n+1}.$$

**Example 7.** Find an iterative equation for the sequence of partial sums of the series

$$\sum_{k=1}^{N} \ln\left(1 + \frac{1}{k}\right).$$

Then find an exact solution.

$$S_1 = \ln\left(1 + \frac{1}{1}\right)$$

$$S_2 = \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) = S_1 + \ln\left(1 + \frac{1}{2}\right)$$

$$S_3 = \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right)$$

$$= S_2 + \ln\left(1 + \frac{1}{3}\right)$$

ADD $(n+1)^{TH}$ TERM

$$\vdots$$

$$S_{n+1} = S_n + \ln\left(1 + \frac{1}{n+1}\right)$$

$$S_N = \sum_{k=1}^{N} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{N} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{N} \ln(k+1) - \ln(k)$$

$$= \underset{k=1}{\left[\cancel{\ln(2)} - \underbrace{\ln(1)}_{0}\right]} + \underset{k=2}{\left[\cancel{\ln(3)} - \cancel{\ln(2)}\right]} + \underset{k=3}{\left[\cancel{\ln(4)} - \cancel{\ln(3)}\right]}$$

$$\ldots + \underset{k=N}{\left[\ln(N+1) - \cancel{\ln(N)}\right]}$$

"TELESCOPING SUM"

$$\boxed{S_N = \ln(N+1)}$$

**Example 8.** Find an iterative equation for the sequence of partial sums of the series

$$\sum_{k=1}^{N} k.$$

Then find an exact solution.

$\rightsquigarrow S_{n+1} = S_n + (n+1)\ ,\quad S_0 = 0$

Solution: $x_0 = 0$

$x_1 = 1$

$x_2 = 3$

$x_3 = 6$

$\vdots$

Exact Sol'n: $F(n) = \sum_{k=1}^{n} k$

$F(n) = 1+2+3+\dots+(n-2)+(n-1)+n = \boxed{(n+1)\frac{n}{2}}$ if $n$ is even

$= 1 + 2 + \dots + \frac{n+1}{2} + \dots + (n-1) + n$ if $n$ is odd

$= (n+1)\frac{n-1}{2} + \frac{n+1}{2} = \frac{n+1}{2}(n-1+1) = \boxed{\frac{n(n+1)}{2}}$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

Find an iterative equation for the sequence of partial sums of the series

$$\sum_{k=1}^{N} k^2.$$

Then find an exact solution.
Hint:

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1, \qquad k = 0, 1, 2, 3, \dots$$

$$
\begin{array}{lrcl}
k=0 & 1^3 &=& 0^3 + 3\cdot 0^2 + 3\cdot 0 + 1 \\
k=1 & 2^3 &=& 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1 \\
k=2 & 3^3 &=& 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1 \\
\vdots & \vdots && \\
k=K & + \ (K+1)^3 &=& K^3 + 3\cdot K^2 + 3\cdot K + 1
\end{array}
$$

$$\sum_{n=1}^{K+1} n^3 = \sum_{n=0}^{K} n^3 + 3\sum_{n=0}^{K} n^2 + 3\sum_{n=0}^{K} n + K+1$$

$$\sum_{n=1}^{K+1} n^3 = \sum_{n=1}^{K} n^3 + 3\boxed{\sum_{n=1}^{K} n^2} + 3\sum_{n=1}^{K} n + K+1$$

SOLVE FOR THIS.

$$\underbrace{\sum_{n=1}^{K+1} n^3 - \sum_{n=1}^{K} n^3} - 3\sum_{n=1}^{K} n - (K+1) = 3\sum_{n=1}^{K} n^2$$

$$(K+1)^3 - 3\,\frac{K(K+1)}{2} - (K+1) = 3\sum_{n=1}^{K} n^2$$

$$\frac{(K+1)}{2}\left[2(K+1)^2 - 3K - 2\right] = 3\sum_{n=1}^{K} n^2$$

$$2(K^2+2K+1) - 3K - 2$$

$$2K^2 + 4K + \cancel{2} - 3K - \cancel{2}$$

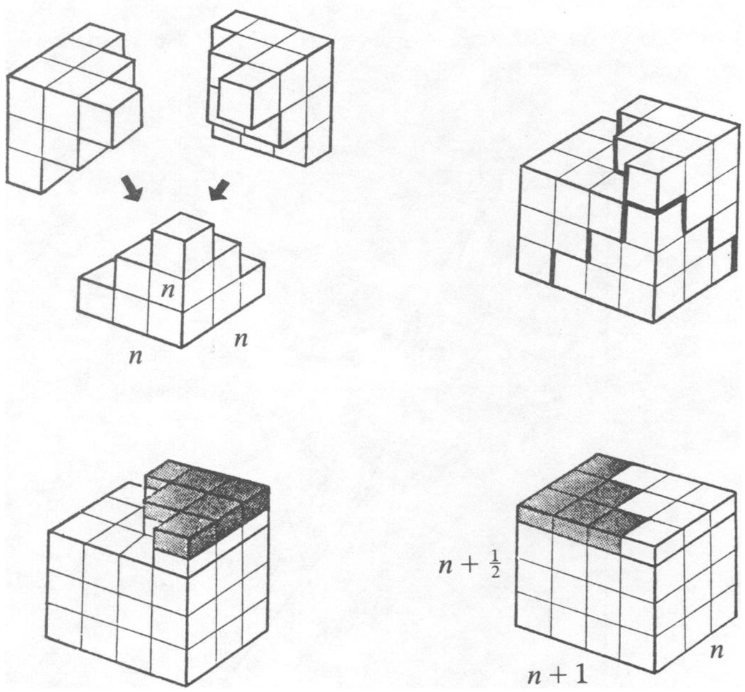
$$\frac{(k+1)}{2}\left[2k^2+k\right] = 3\sum_{n=1}^{k} n^2$$

$$\frac{(k+1)\,k\,[2k+1]}{2} = 3\sum_{n=1}^{k} n^2$$

$$\boxed{\frac{(k+1)\,k\,[2k+1]}{6} = \sum_{n=1}^{k} n^2}$$

**Proof without words:**
**Sum of squares**

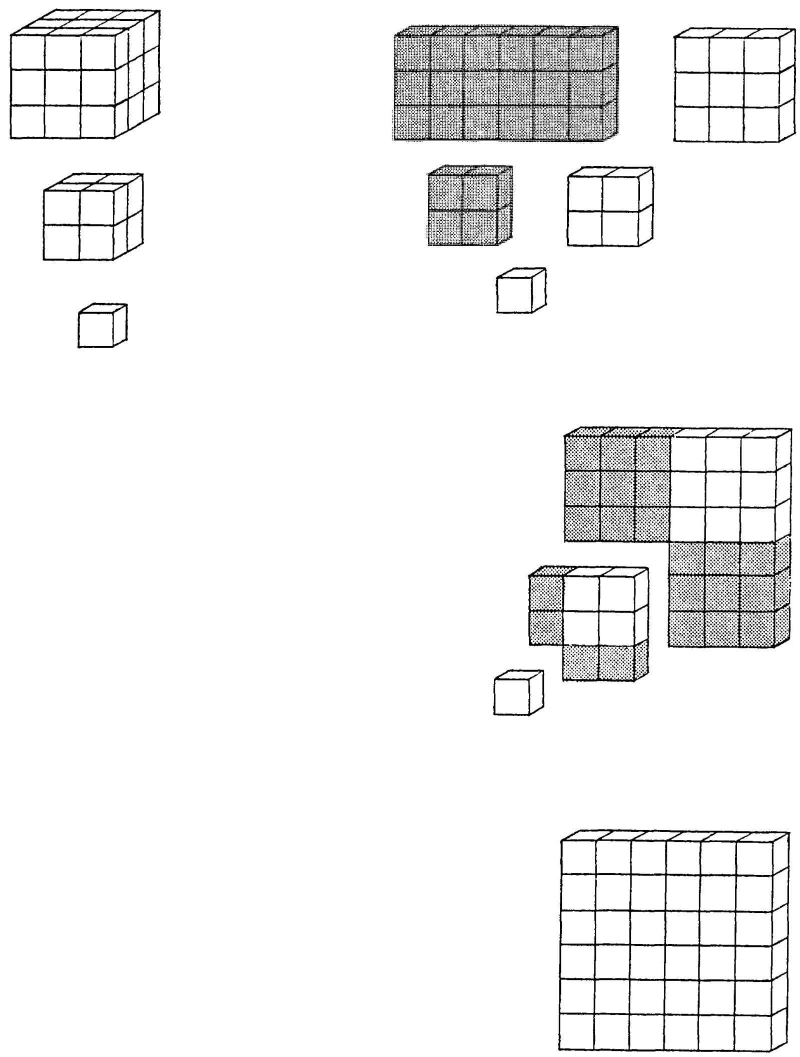
$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{3}n(n+1)\left(n+\tfrac{1}{2}\right)$$




—Man-Keung Siu
University of Hong Kong

## Proof without words: Sum of cubes

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$



—ALAN L. FRY
University of North Carolina
at Greensboro