Lecture 02: 2.1 Some linear models, 2.2 Linear Equations and their solutions

Definition 1. Given a model/iterative equation

$$x_{n+1} = f(x_n),$$

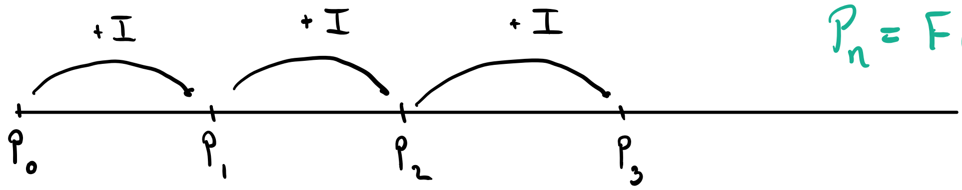
RECURSIVE

an exact solution to the model is a formula for calculating x_n without having to first calculate $x_1, x_2, x_3, \ldots, x_{n-1}$.

$x_n = F(n)$ EXACT

Example 1. Can you find an *exact solution* to the simple interest model $P_{n+1} = P_n + I$?

$P_n = F(n)$



P_0

$P_1 = P_0 + I$

$P_2 = P_1 + I = P_0 + I + I$

$P_3 = P_2 + I = P_0 + I + I + I$

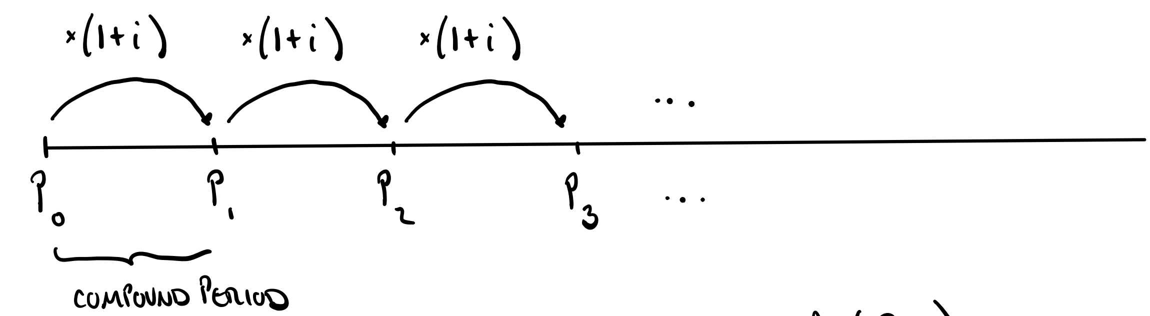
$$\boxed{P_n = P_0 + In}$$

Compound interest

If a principal investment of P_0 dollars earns an annual interest rate r compounded m times per year, then the investment earns an interest rate

$$i = \frac{r}{m}$$

per compound period, where m compound periods equals 1 year.



MODEL: $P_{n+1} = f(P_n)$

$P_n = (1+i)P_{n-1}$ $\longleftrightarrow$ $\boxed{P_{n+1} = (1+i)P_n}$

P_0

$P_1 = (1+i)P_0$

$P_2 = (1+i)P_1 = (1+i)(1+i)P_0 = (1+i)^2 P_0$

$P_3 = (1+i)P_2 = (1+i)(1+i)^2 P_0 = (1+i)^3 P_0$

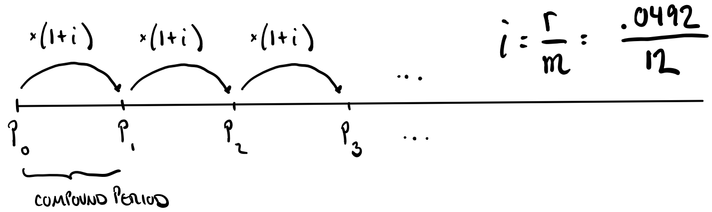
$P_n = (1+i)^n P_0$

EXACT SOL'N

The value of the investment P_n after n compound periods evolves according to the formula

$$P_{n+1} = (1+i)P_n. \qquad \text{(Compond Interest Model)}$$

Example 2. Suppose \$1800 is invested in an account that earns 4.92% annual interest compounded monthly. How long will it take for the investment to grow to a value of 2000?



$$i = \frac{r}{m} = \frac{.0492}{12}$$

$$P_0 = 1800$$

$$\xrightarrow{\times 1.0041} P_1 =$$

$$\xrightarrow{\times 1.0041} P_2 =$$

$$\vdots$$

$$P_{26} = 2002. \qquad \text{26 MONTHS.}$$

$$\underbrace{P_n}_{2000} = (1+i)^n \underbrace{P_0}_{1800} \longrightarrow$$

$$2000 = (1.0041)^n \, 1800 \qquad \text{SOLVE FOR } n$$

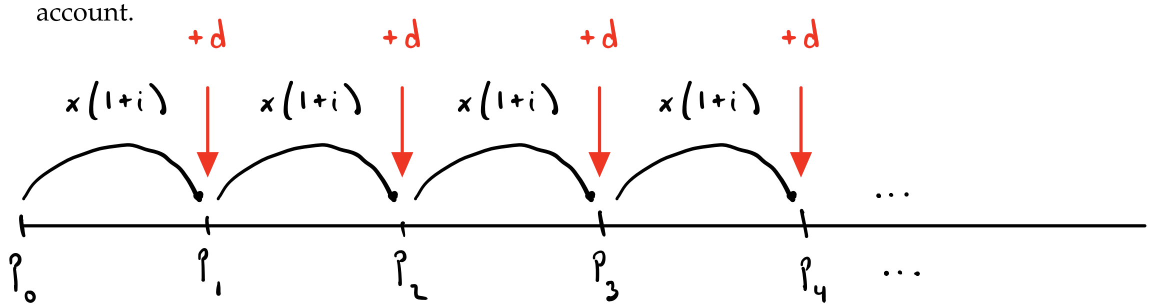
$$\frac{2000}{1800} = (1.0041)^n$$

$$n = \log_{1.0041}\left(\frac{2000}{1800}\right)$$

$$= \frac{\log\left(\frac{2000}{1800}\right)}{\log(1.0041)} \qquad \text{(ROUND UP)}$$

Annuity savings model

An initial investment of P_0 dollars is made into an account earning an annual interest rate r compounded m times per year, i.e. interest rate $i = r/m$ per compound period. Additionally, at the end of each compound period, a deposit of d dollars is made into the same account.



MODEL: $$\boxed{P_{n+1} = P_n(1+i) + d}$$

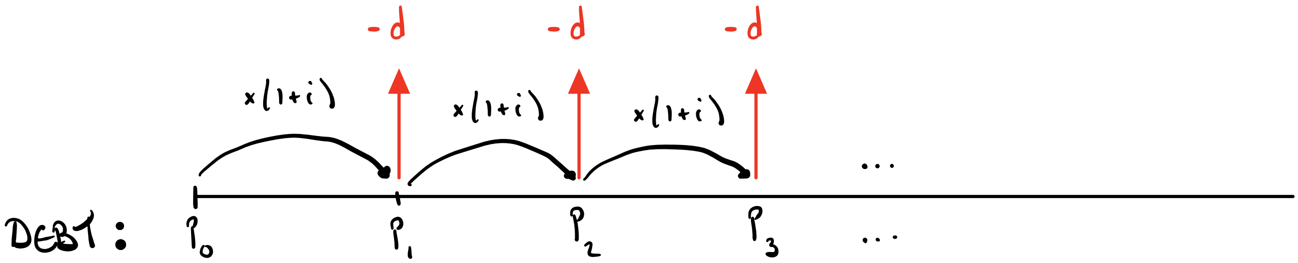
EXACT SOL'N ?

The account balance P_n after n compound periods evolves according to the formula

$$P_{n+1} = (1+i)P_0 + d \qquad \text{(Annuity Savings Model)}$$

Loan payment model

An amount of P_0 dollars in borrowed at an annual interest rate r compounded m times per year, i.e. interest rate $i = r/m$ per compound period. Additionally, at the end of each compound period, a payment of d dollars is made toward the balance of the loan.



$$\boxed{P_{n+1} = P_n(1+i) - d}$$

The balance of the loan P_n after n compound periods evolves according to the formula

$$P_{n+1} = (1+i)P_0 - d \qquad \text{(Loan Payment Model)}$$

Example 3. Construct a loan payment model that satisfies

$$P_0 = 10,000 \qquad P_1 = 9800 \qquad P_2 = 9594.$$

$$P_{n+1} = (1+i)\,P_n - d$$

$n=0$: $P_1 = (1+i)P_0 - d$

$n=1$: $P_2 = (1+i)P_1 - d$

$$9800 = (1+i)\,10,000 - d$$
$$9594 = (1+i)\,9800 - d$$

$$9800 = (1+i)\,10,000 - d$$
$$-9594 = -(1+i)\,9800 + d \qquad \text{(ELIMINATION)}$$

$$206 = (1+i)\,200$$

$$\frac{206}{200} = 1+i \implies i = \frac{6}{200} = .03$$

$$9800 = (1+i)\,10,000 - d$$
$$9800 = (1.03)\,10,000 - d$$
$$d = 10,300 - 9800 = 500$$

$i = .03$

$d = 500$

$$\boxed{P_{n+1} = (1.03)\,P_n - 500}$$

Population Models

Linear population model

Suppose there is a population of a certain specied with an initial population P_0.

- Assume the population is isolated
  - The population increases only through births
  - The population decreases only through deaths
- Assume the number of births and deaths each year (generation) is proportional to the size of the population. RULE OF NATURAL GROWTH

$$P_{n+1} = P_n - \underbrace{dP_n}_{\substack{\#\text{ DEATH} \\ \sim P_n}} + \underbrace{bP}_{\substack{\#\text{ BIRTHS} \\ \sim P_n}}, \qquad \begin{array}{l} 0 \le b \\ 0 \le d \le 1 \end{array}$$

$$P_{n+1} = P_n(\underbrace{1-d+b}_{r}) \Rightarrow P_{n+1} = rP_n$$

The population P_n after n years (generations) evolves according to the formula

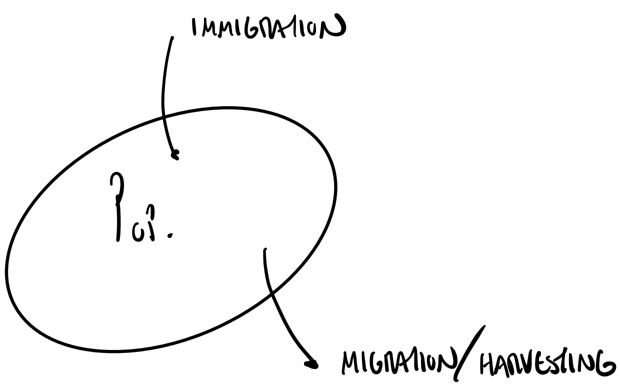
$$P_{n+1} = rP_n, \qquad \text{(Linear Population Model)}$$

where r is the *growth rate* of the population.

Linear immigration/migration/harvestng model

Now assume that the population is *not* isolated.

- Individuals may *immigrate* into the population from other populations
- Individuals may *migrate* out of the population to other populations
- Individuals may be *harvested* by other species



$$P_{n+1} = rP_n + \underbrace{c_1 - c_2}_{k}$$

$$\boxed{P_{n+1} = rP_n + k}$$

The population P_n after n years (generations) evolves according to the formula

$$P_{n+1} = rP_n + k, \quad \text{(Linear Population Model with Immigration/Migration/Harvesting)}$$

where r is the *growth rate* of the population and k is the net number of individuals gained ($k > 0$) or lost ($k < 0$) through immigration, migration, and harvesting.

Example 4. Suppose a population of deer, initially 275,000, has a birth rate of 8% and a death rate of 6.5% per generation. How many deer should the department of wildlife allow people to hunt each year so that the population remains constant?

$$P_{n+1} = r P_n - \underbrace{k}_{\text{HUNTING}}$$

$P_0 = 275{,}000$

$P_1 = 275{,}000$ (SAME)

$\vdots$

275,000 FOREVER.

$$r = (1 + .08 - .065) = 1.015$$

$$P_{n+1} = 1.015 P_n - k$$

$$275{,}000 = (1.015)\,275{,}000 - k$$

$$k = (1.015)\,275{,}000 - 275{,}000 = 275{,}000\,(1.015 - 1)$$

$$= 4{,}125$$