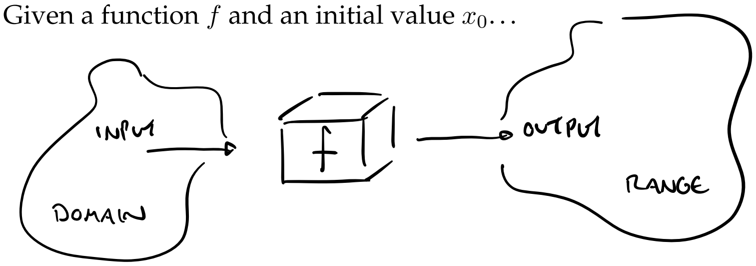
Lecture 01: Welcome, Ch1 Introduction to discrete dynamical systems, 2.1 Some linear models

Discrete dynamical systems

Given a function f and an initial value x_0...



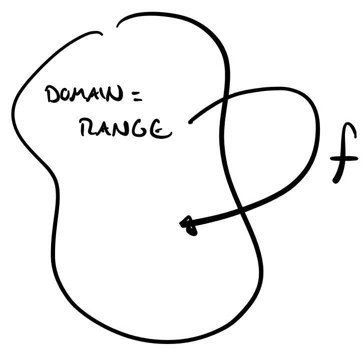
$\text{TAN}(a) = \frac{\pi}{2}$

$\text{TAN}(\text{TAN}(a)) = \text{TAN}\,\frac{\pi}{2} = \text{UND.}$

$$x_0 \xrightarrow{f} f(x_0) = x_1 \xrightarrow{f} f(f(x_0)) = f(x_1) = x_2 \xrightarrow{f} f(f(f(x_0))) = f(x_2) = x_3$$

... CONTINUE ...

$\Rightarrow$ SEQUENCE : $x_0, x_1, x_2, x_3, \ldots$



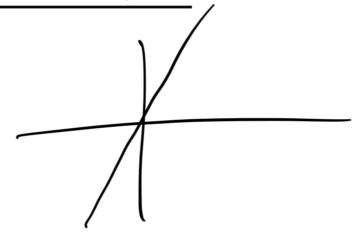
NOTE: THIS PROCESS REQUIRES THAT RANGE OF f IS SUBSET OF DOMAIN OF f.

The function f determines how to values x_n evolve in *discrete* time steps.

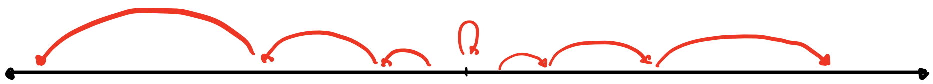
$$\underbrace{x_{n+1} = f(x_n)}_{\text{BREAD \& BUTTER}} \qquad \text{(Iterative Equation)}$$

"FLATLAND"

Example 1. Sketch the phase portraits for the following functions f.



i. $f(x) = 2x$ FUNCTIONS = MAPS

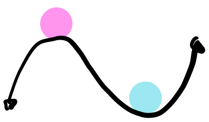


0

$f(0)=0$

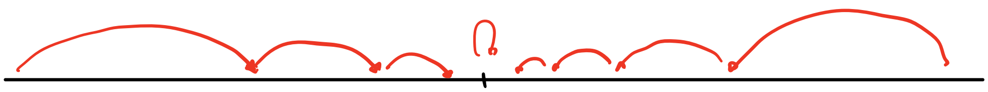
0 is "FIXED POINT" UNSTABLE / REPELLER

UNSTABLE



STABLE

ii. $f(x) = \frac{1}{2}x$



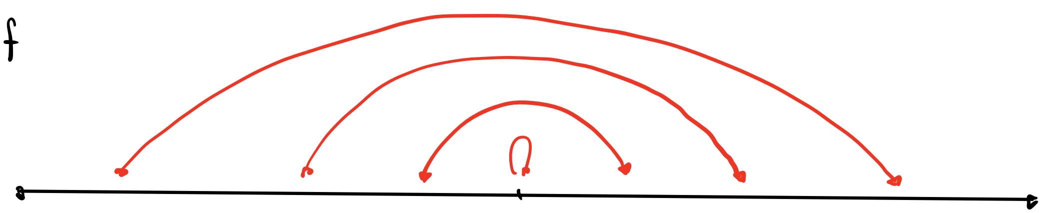
0

$f(0)=0$

0 is "FIXED POINT" STABLE / ATTRACTOR

iii. $f(x) = -x$

$f^{-1} = f$



0

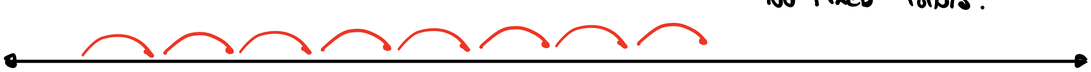
NEITHER STABLE NOR UNSTABLE

$\boxed{\text{CHALLENGE: } f(x) = \frac{1}{x} \text{ ?}}$

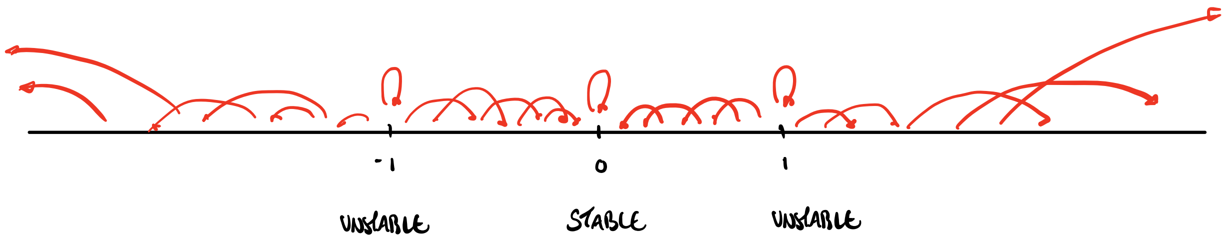
iv. $f(x) = x + c$

$c > 0$

NO FIXED POINTS.

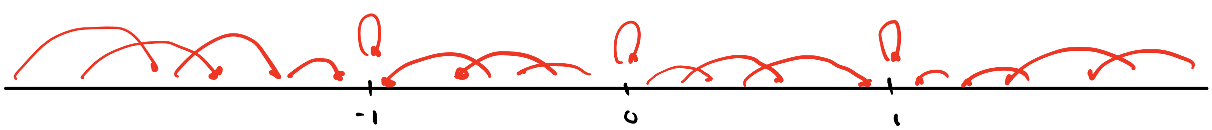


v. $f(x) = x^3$



INVERSE

vi. $f(x) = \sqrt[3]{x}$



Modelling reality

Example 2. A bouncy ball is dropped from a height of 1m above the ground. Every time the ball bounces, it loses 10% of its energy. Let s_n equal the maximum height of the ball after n bounces.

MAX HEIGHT PROPORTIONAL TO ENERGY

$$s_0 = 1$$
$$s_1 = .9$$
$$s_2 = .81$$
$$s_3 = .729$$

$\times .9$ $\times .9$ $\times .9$

$\Rightarrow$

BOUNCING BALL MODEL

$$\boxed{S_{n+1} = r S_n, \quad r > 0. \quad n = 0, 1, 2, 3, \dots}$$

e.g. $r = .9$

Example 3. An object in freefall experiences constant accellaration. Let v_n equal the velocity of a freefalling object n seconds after being dropped form a very high point.

$$V_0 = 0$$
$$V_1 = g$$
$$V_2 = 2g$$
$$V_3 = 3g$$
$$\vdots$$

$+g$ (CONSTANT) $+g$ $+g$

ITERATIVE EQUATION
- DESCRIBES HOW A QUANTITY EVOLVES

$$\boxed{V_{n+1} = V_n + g}$$

$$\left(V_n = ng \right)$$

EXACT SOLUTION

Example 4. Refine the previous model to account for air resistance.

$$V_{n+1} = \boxed{V_n + g - pV_n}$$

AIR PUSHES AGAINST FALLING OBJECT
AN AMOUNT PROPORTIONAL TO VELOCITY

$$V_{n+1} = \boxed{f(V_n)}$$

$$f(x) = x + g - px = (1-p)x + g$$

Financial models

Simple interest

If a principal investment of P_0 dollars earns an annual simple interest rate r, then the interest earned each year is

$$I = rP_0.$$

Let P_n equal the value of the investment after n years.

P_0

$P_1 = P_0 + I$

$P_2 = P_1 + I$

$P_3 = P_2 + I$

$+I$ $+I$ $+I$

$P_{n+1} = P_n + I.$

The value of the investment after n years evolves according to the formula

$$P_{n+1} = P_n + I. \qquad \text{(Simple Interest Model)}$$