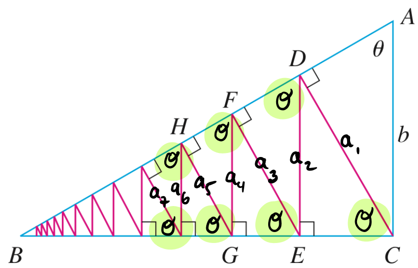
Written Homework

1. A right triangle ABC is given with $\angle A = \theta$ and $|\overline{AC}| = b$. The line segment $\overline{CD}$ is perpendicular to $\overline{AB}$, the line segment $\overline{DE}$ is perpendicular to $\overline{BC}$, and so on, as shown in the figure. Find the total length of all the perpendiculars

$$|\overline{CD}| + |\overline{DE}| + |\overline{EF}| + |\overline{FG}| + \dots$$

in terms of b and θ. Hint: Use similar triangles and trigonometry to form a geometric series.



ALL RIGHT TRIANGLES SHOWN HAVE THE SAME 3 ANGLES : $\frac{\pi}{2}, \theta, \frac{\pi}{2} - \theta$ (MUST ADD TO π)

$a_1 = b \sin\theta$

$a_2 = a_1 \sin\theta = (b\sin\theta)\sin\theta = b\sin^2\theta$

$a_3 = a_2 \sin\theta = (b\sin^2\theta)\sin\theta = b\sin^3\theta$

ETC.

THUS $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b\sin^n\theta$ OR $\sum_{n=1}^{\infty} (b\sin\theta)(\sin\theta)^{n-1}$ GEOMETRIC SERIES

THIS IS A GEOMETRIC SERIES $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ IF $|r| < 1$

HERE $a = b\sin\theta$ & $r = \sin\theta$.

SINCE $0 < \theta < \frac{\pi}{2}$, $|\sin\theta| < 1$ & SO

$$\sum_{n=1}^{\infty} (b\sin\theta)(\sin\theta)^{n-1} = \boxed{\frac{b\sin\theta}{1-\sin\theta}}$$

2. For each of the following series, determine if the series converges absolutely, converges conditionally, or diverges. Justify your answers.

(a) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

INTEGRAL TEST : $f(x) = \frac{1}{x\sqrt{\ln x}}$ IS POSITIVE & DECREASING FOR $x \geq 2$, AND

$f(n) = \frac{1}{n\sqrt{\ln n}}$.

$$\int_{2}^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\,du$$ DIVERGES BY p-TEST $\left(p = \frac{1}{2} < 1\right)$.

$\therefore$ $\boxed{\text{SERIES DIVERGES BY INTEGRAL TEST.}}$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

ALTERNATING SERIES TEST : TERMS ALTERNATE POSITIVE/NEGATIVE.

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} : \frac{\infty}{\infty} \overset{(\text{L'Ho.})}{=} \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$$

$\therefore$ SERIES CONVERGES BY ALTERNATING SERIES TEST.

NOW WE CHECK FOR ABSOLUTE CONVERGENCE:

$$\left| (-1)^n \frac{\ln n}{\sqrt{n}} \right| = \frac{\ln n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} \quad \text{FOR } n \geq 3.$$

SINCE $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES BY p-TEST $\left(p = \frac{1}{2} < 1\right)$,

THE SERIES DOES NOT CONVERGE ABSOLUTELY. THEREFORE, IT $\boxed{\text{CONVERGES CONDITIONALLY.}}$

(c) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

NOTE THAT
$$(\ln n)^{\ln n} = e^{\ln\left[(\ln n)^{\ln n}\right]} = e^{\ln n \ln(\ln n)}$$
$$= \left[e^{\ln n}\right]^{\ln(\ln n)}$$
$$= n^{\ln(\ln n)}$$

FURTHERMORE, $\dfrac{1}{n^{\ln(\ln n)}} \le \dfrac{1}{n^2}$ EVENTUALLY

WHEN $\ln(\ln n) \ge 2$

$\ln n \ge e^2$

$n \ge e^{e^2}$ ($\approx$ 1619, FYI)

$\therefore$ SINCE $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ CONVERGES BY p-TEST WITH $p = 2 > 1$,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln(\ln n)}}$ ALSO CONVERGES BY DIRECT COMPARISON TEST.

SINCE ALL TERMS ARE ALREADY POSITIVE, $\boxed{\text{THE SERIES CONVERGES ABSOLUTELY.}}$

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}$

RATIO TEST: $\displaystyle\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{\cancel{n+1}}(n+1)\cancel{!}}{\cancel{5}\cdot\cancel{8}\cdot\cancel{11}\cdot\ldots\cdot\cancel{(3n+2)}(3(n+1)+2)} \cdot \frac{\cancel{5}\cdot\cancel{8}\cdot\cancel{11}\cdot\ldots\cdot\cancel{(3n+2)}}{\cancel{2^n}\, n\cancel{!}}$

$\displaystyle= \lim_{n\to\infty} \frac{2n+2}{3n+5} = \frac{2}{3} < 1.$

HENCE THE $\boxed{\text{SERIES CONVERGES ABSOLUTELY BY RATIO TEST.}}$

3. For each of the following power series, find the radius of convergence and the interval of convergence.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$

RATIO TEST: $\displaystyle\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{\cancel{n+1}}}{2^{\cancel{n+1}}\ln(n+1)} \cdot \frac{\cancel{2^n}\ln n}{\cancel{(x+2)^n}}\right|$

(ROOT TEST ALSO WORKS)

$\displaystyle= \frac{|x+2|}{2}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} : \frac{\infty}{\infty}$

$\displaystyle\overset{(\text{L'HO})}{=} \frac{|x+2|}{2}\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x+2|}{2}$

THE SERIES CONVERGES WHEN $\rho < 1$, i.e. WHEN $\dfrac{|x+2|}{2} < 1$.

THIS GIVES $|x+2| < 2$ $\Rightarrow$ $\boxed{\text{RADIUS OF CONVERGENCE IS } R=2}$

(CENTER OF POWER SERIES IS -2)

$-2 < x+2 < 2$

$-4 < x < 0$

NOW WE JUST NEED TO CHECK ENDPOINTS TO FIND INTERVAL OF CONVERGENCE.

$x=-4$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$.

SINCE $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$, THIS SERIES CONVERGES BY ALTERNATING SERIES TEST.

$x=0$: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n}$ WHICH DIVERGES (HARMONIC SERIES).

HENCE THIS SERIES DIVERGES BY DIRECT COMPARISON TEST.

THEREFORE, $\boxed{\text{INTERVAL OF CONVERGENCE IS } [-4,0)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

ROOT TEST: $\displaystyle\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{(2x-1)^n}{5^n \sqrt{n}} \right|^{1/n} = \lim_{n\to\infty} \left| \frac{(2x-1)}{5\sqrt{n^{1/n}}} \right|$

$$= \frac{2|x-\frac{1}{2}|}{5} \cdot \frac{1}{\sqrt{\underbrace{\lim_{n\to\infty} n^{1/n}}_{\substack{\text{IND. POWER } \infty^0 \\ \text{L'HO RULE} \Rightarrow \text{ LIMIT IS } 1.}}}} = \frac{2}{5}|x-\tfrac{1}{2}|$$

SERIES CONVERGES WHEN $\rho = \frac{2}{5}|x-\frac{1}{2}| < 1$

THAT IS, $|x-\frac{1}{2}| < \frac{5}{2}$ $\Rightarrow$ $\boxed{\text{RADIUS OF CONVERGENCE } R=\frac{5}{2}}$

(CENTER OF POWER SERIES IS $\frac{1}{2}$)

$-\frac{5}{2} < x-\frac{1}{2} < \frac{5}{2}$

$-2 < x < 3$ NOW WE CHECK ENDPOINTS FOR CONVERGENCE.

$x = -2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ CONVERGES BY ALT. SERIES TEST.

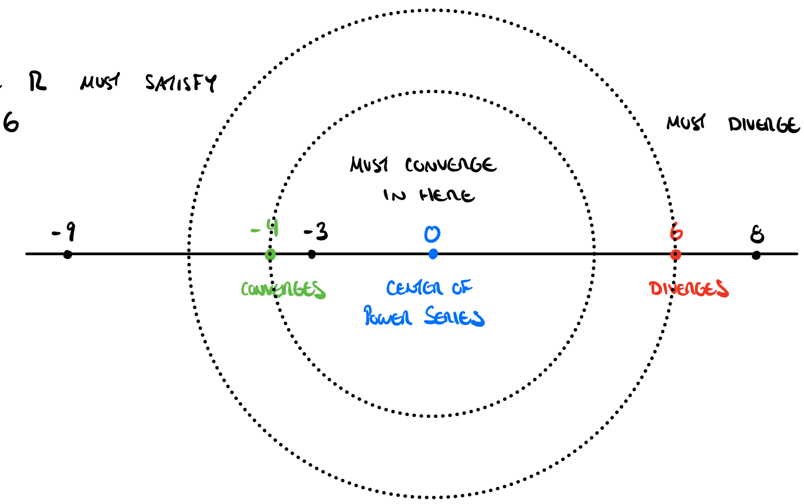
$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES BY p-TEST.

THEREFORE, $\boxed{\text{INTERVAL OF CONVERGENCE IS } [-2, 3)}$

4. Suppose that the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. Then which of the following series must converge?

RADIUS OF CONVERGENCE R MUST SATISFY
$4 \leq R \leq 6$

MUST DIVERGE OUT HERE



(a) $\displaystyle\sum_{n=0}^{\infty} c_n$ $x = 0$ CONVERGE ✓

(b) $\displaystyle\sum_{n=0}^{\infty} c_n 8^n$ $x = 8$ DIVERGE ✗

(c) $\displaystyle\sum_{n=0}^{\infty} c_n(-3)^n$ $x = -3$ CONVERGE ✓

(d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 9^n$ $x = -9$ DIVERGE ✗