Written Homework

§8.1-3 and §10.1-4

Solutions

1. (8 points) Find the length of the curve $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, $1 \le x \le 4$.

$$f(x) = \frac{1}{12}x^3 + \frac{1}{x}$$

$$f'(x) = \frac{1}{4}x^2 - \frac{1}{x^2}$$

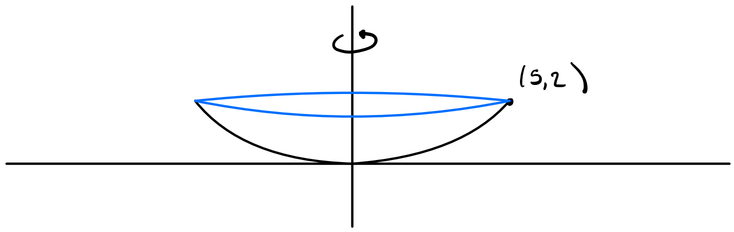
$$L = \int_1^4 \sqrt{1 + f'(x)^2}\, dx = \int_1^4 \sqrt{1 + \left(\frac{1}{4}x^2 - \frac{1}{x^2}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4}}\, dx$$

$$= \int_1^4 \sqrt{\frac{1}{16}x^4 + \frac{1}{2} + \frac{1}{x^4}}\, dx = \int_1^4 \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2}\, dx = \int_1^4 \frac{1}{4}x^2 + \frac{1}{x^2}\, dx$$

$$= \frac{1}{12}x^3 - \frac{1}{x}\Big|_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{16}{3} - \frac{1}{4} - \frac{1}{12} + 1$$

$$= \frac{64 - 3 - 1 + 12}{12} = \frac{72}{12} = \boxed{6}$$

2. (10 points) A group of engineers is building a parabolic satelite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 10 ft diameter and a maximum depth of 2 ft, find the value of a and the surface area of the dish.



$(5,2)$ is on $y = ax^2 \Rightarrow 2 = a(5)^2 \Rightarrow \boxed{a = \frac{2}{25}}$

$y = \frac{2}{25}x^2$

Surface Area $S = \int_0^5 2\pi x \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$

$= \int_0^5 2\pi x \sqrt{\left(\frac{4}{25}x\right)^2 + 1}\, dx$

$= \int_0^5 2\pi x \sqrt{\frac{16}{625}x^2 + 1}\, dx$

Let $u = \frac{16}{625}x^2 + 1$

$du = \frac{32}{625} x\, dx$

$\sim \frac{625}{32} \cdot 2\pi \int_1^{41/25} \sqrt{u}\, du$

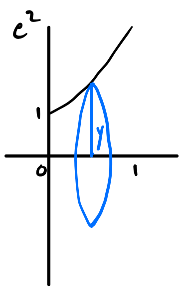
$= \frac{625\pi}{16} \left[\frac{2}{3} u^{3/2} \right]_1^{41/25}$

$$\boxed{= \frac{625\pi}{24}\left(\left(\frac{41}{25}\right)^{3/2} - 1 \right) \approx 90.0119 \text{ ft}^2}$$

3. Setup (but do not evaluate) two integrals – one with respect to x and one with respect to y – for the surface area of the solid obtained by rotation the curve

$$y = e^{2x}, \quad 0 \le x \le 1$$

(a) (8 points) about the x-axis.



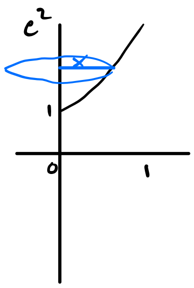
$$y = e^{2x} \iff x = \frac{1}{2}\ln y$$

$$\frac{dy}{dx} = 2e^{2x} \qquad \frac{dx}{dy} = \frac{1}{2y}$$

$$A = \int 2\pi y \, ds = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad \int_1^{e^2} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \boxed{\int_0^1 2\pi e^{2x} \sqrt{1 + (2e^{2x})^2}\, dx} \quad \text{or} \quad \boxed{\int_1^{e^2} 2\pi y \sqrt{1 + \left(\frac{1}{2y}\right)^2}\, dy}$$

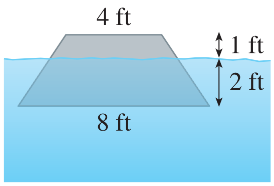
(b) (8 points) about the y-axis.

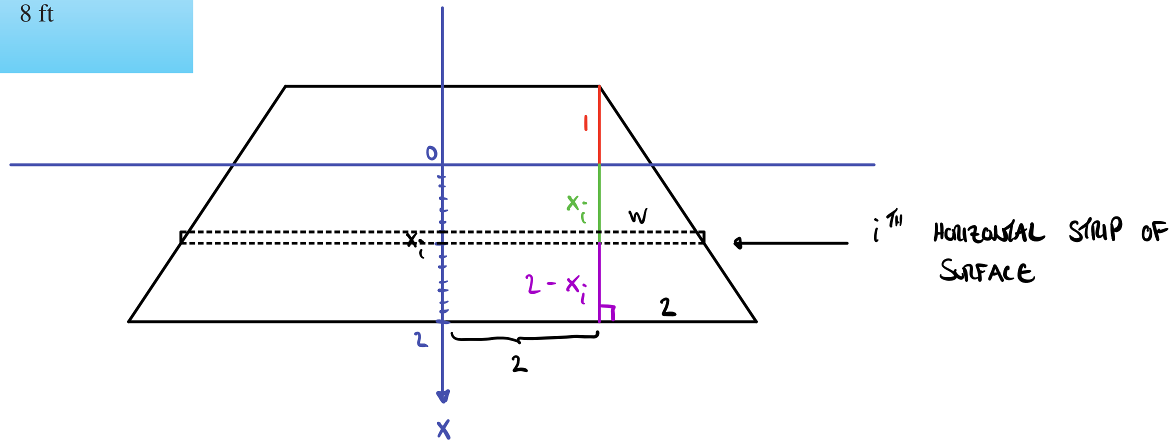


$$A = \int 2\pi x \, ds = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad \int_1^{e^2} 2\pi \frac{1}{2}\ln y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \boxed{\int_0^1 2\pi x \sqrt{1 + (2e^{2x})^2}\, dx} \quad \text{or} \quad \boxed{\int_1^{e^2} \pi \ln y \sqrt{1 + \left(\frac{1}{2y}\right)^2}\, dy}$$

4. (10 points) A vertical plate in the shape of a trapezoid is partially submerged in water as shown below. Calculate the hydrostatic force against one side of the plate.





SIMILAR TRIANGLES : $\dfrac{W}{1+x_i} = \dfrac{2}{3} \Rightarrow W = \dfrac{2}{3}(1+x_i)$

$4 + 2W$

THE FORCE ON THE i^{TH} HORIZONTAL STRIP

$= \delta(\text{AREA})(\text{DEPTH}) = 62.5\left[2(2+W)\,\Delta x\right] x_i$

$4 + \dfrac{4}{3}(1+x_i)$

$= 62.5 \times 2\left(2 + \dfrac{2}{3}(1+x_i)\right) x_i\,\Delta x$

$4 + \dfrac{4}{3} + \dfrac{4}{3}x_i$

$= \dfrac{250}{3}\left(4x_i + x_i^2\right)\Delta x$

$\dfrac{16}{3} + \dfrac{4}{3}x_i$

$\therefore$ TOTAL FORCE $\approx \displaystyle\sum_{i=1}^{n} \frac{250}{3}\left(4x_i + x_i^2\right)\Delta x$

$\dfrac{4}{3}\left(4 + x_i\right)$

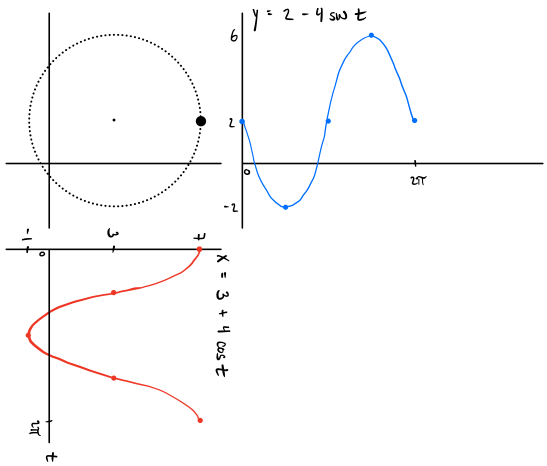
TOTAL FORCE $= \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{250}{3}\left(4x_i + x_i^2\right)\Delta x$

$= \dfrac{250}{3}\displaystyle\int_0^2 4x + x^2\,dx = \frac{250}{3}\left[2x^2 + \frac{1}{3}x^3\right]_0^2$

$= \dfrac{250}{3}\left[8 + \dfrac{8}{3}\right] = \left(\dfrac{250}{3}\right)\left(\dfrac{32}{3}\right) = \boxed{\dfrac{8000}{9} \approx 889 \text{ LBS.}}$

5. Find parametric equations for the path of a particle that moves along the circle with center $(3, 2)$ and radius 4 in the manner described.

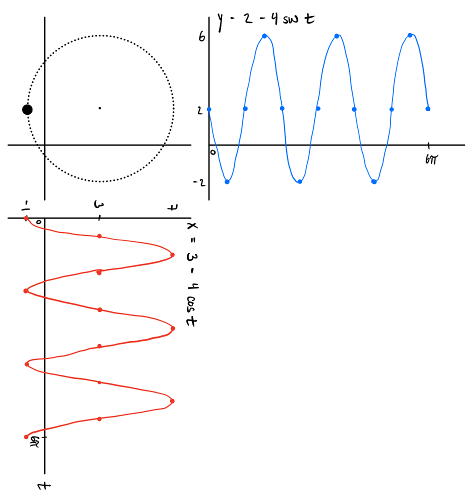
   (a) (6 points) Once around clockwise starting at $(7, 2)$



$$x = 3 + 4\cos t$$

$$y = 2 - 4\sin t, \quad 0 \le t \le 2\pi$$

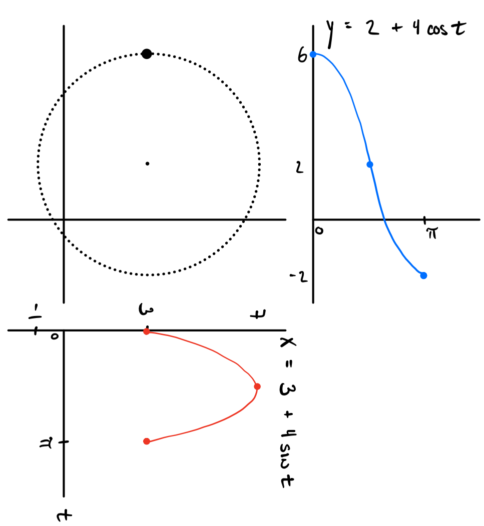
   (b) (6 points) Three times around counterclockwise starting at $(-1, 2)$



$$x = 3 - 4\cos t$$

$$y = 2 - 4\sin t, \quad 0 \le t \le 6\pi$$

   (c) (6 points) Halfway around clockwise starting at $(3, 6)$



$$x = 3 + 4\sin t$$

$$y = 2 + 4\cos t, \quad 0 \le t \le \pi$$

6. (8 points) An ellipse is described by the parametric equations

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \leq t \leq 2\pi,$$

where a and b are the semi-major and semi-minor axes. Setup (but do not evaluate) a definite integral for the circumference of the ellipse.

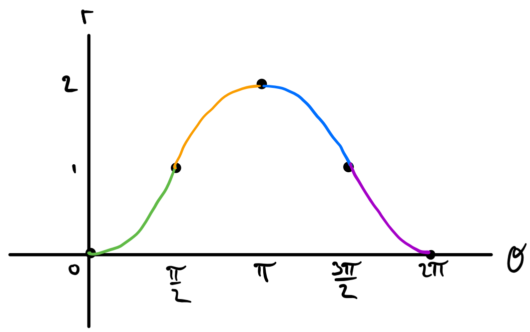
$x' = -a\sin t$

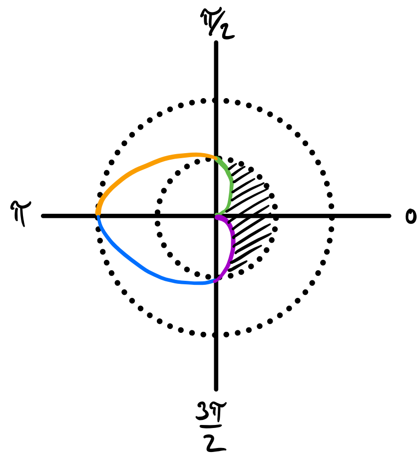
$y' = b\cos t$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

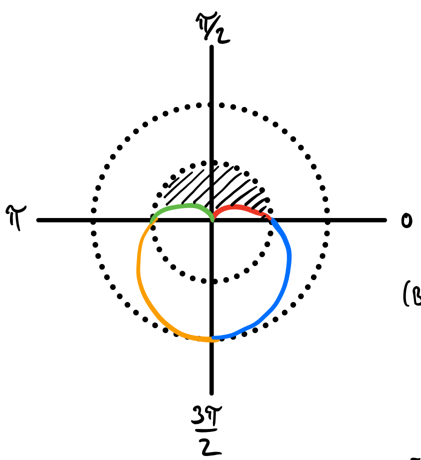
$$L = \boxed{\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt}$$

7. (a) (4 points) Sketch the polar graph $r = 1 - \cos\theta$.





(b) (8 points) Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

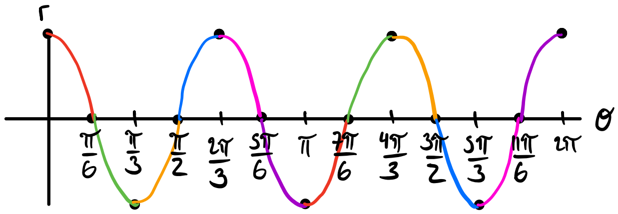


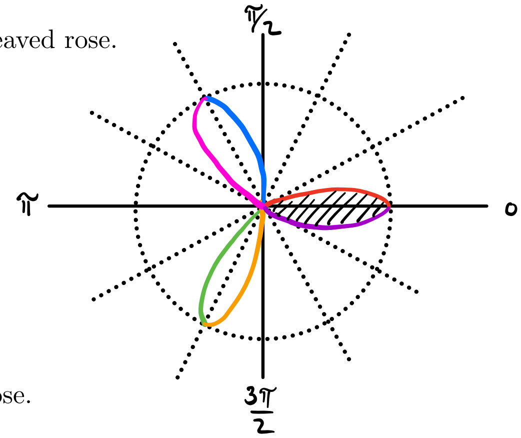
AREA $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1)^2 - \frac{1}{2}(1-\cos\theta)^2 \, d\theta$

(BY SYMMETRY) $= \int_0^{\pi/2} 1 - (1-\cos\theta)^2 \, d\theta = \int_0^{\pi/2} 2\cos\theta - \cos^2\theta \, d\theta$

$= \left[2\sin\theta - \frac{1}{4}(2\theta + \sin 2\theta) \right]_0^{\pi/2} = \boxed{2 - \frac{\pi}{4}}$

8. (a) (4 points) Sketch the polar graph $r = \cos(3\theta)$, called a three-leaved rose.





(b) (8 points) Find the area inside one petal of the three-leaved rose.

AREA $A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta \, d\theta = \int_{0}^{\pi/6} \frac{1}{2}(1+\cos 6\theta)\, d\theta$

$$= \frac{1}{2}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6} = \boxed{\frac{\pi}{12}}$$

(c) (6 points) Setup an integral for the circumference of one petal of the three-leaved rose (do not evaluate).

$$s = \int_{-\pi/6}^{\pi/6} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$x = r\cos\theta = \cos 3\theta \cos\theta$ $\qquad \frac{dx}{d\theta} = 3\sin 3\theta\cos\theta + \cos 3\theta \sin\theta$

$y = r\sin\theta = \cos 3\theta \sin\theta$ $\qquad \frac{dy}{d\theta} = 3\sin 3\theta \sin\theta + \cos 3\theta\cos\theta$

$$= \boxed{\int_{-\pi/6}^{\pi/6} \sqrt{(3\sin 3\theta\cos\theta + \cos 3\theta\sin\theta)^2 + (3\sin 3\theta\sin\theta + \cos 3\theta\cos\theta)^2}\, d\theta}$$