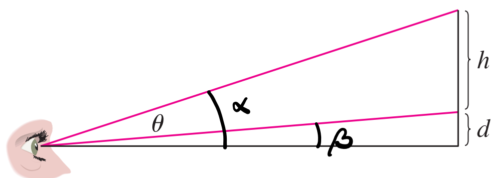
Written Homework 1

§6.6-8, 7.1-4

Solutions

1. (10 points) A TV screen with height h is positioned on a wall so that its lower edge is a distance d above the eye of an observer when seated. How far from the wall should the observer sit to get the best view? That is, how far from the wall should the observer sit so as to maximize the angle θ?



$\theta = \alpha - \beta$

WE MUST MAXIMIZE $\theta = \alpha - \beta = \tan^{-1}\frac{h+d}{x} - \tan^{-1}\frac{d}{x} = f(x)$.

$$f'(x) = \frac{1}{1+\left(\frac{h+d}{x}\right)^2}\cdot\frac{-(h+d)}{x^2} - \frac{1}{1+\left(\frac{d}{x}\right)^2}\cdot\frac{-d}{x^2} = 0$$

$$\frac{-(h+d)}{x^2+(h+d)^2} + \frac{d}{x^2+d^2} = 0$$

$$(x^2+d^2)(h+d) = d\left[x^2+(h+d)^2\right]$$

$$(h+d)x^2 + d^2(h+d) = dx^2 + d(h+d)^2$$

$$hx^2 + (h+d)(d^2 - d(h+d)) = 0$$

$$hx^2 + (h+d)(-dh) = 0$$

$$x^2 = d(h+d) \Rightarrow x = \pm\sqrt{d(h+d)}, \quad x \geq 0$$

$$\boxed{x = \sqrt{d(h+d)}}$$

NOTE: THE HIGHER THE SCREEN IS POSITIONED, AND THE TALLER THE SCREEN IS, THE FURTHER BACK ONE SHOULD SIT.

OUR INTUITION TELLS US THIS IS A MAXIMUM FOR $\theta = f(x)$. 1ST OR 2ND DERIV. TEST COULD BE USED TO VERIFY.

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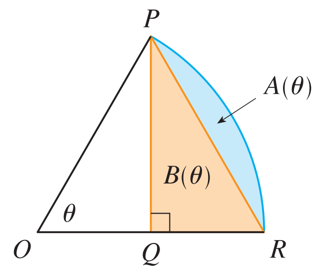
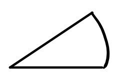
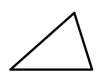
2. (10 points) The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the region between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find the limit

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}.$$



$A(\theta) =$  $-$  $= \frac{\theta}{2\pi} \cdot \pi R^2 - \frac{1}{2} R \cdot R \sin\theta$

$= \frac{R^2}{2}\left(\theta - \sin\theta\right)$

$B(\theta) = \frac{1}{2}\left(R - R\cos\theta\right)\left(R\sin\theta\right) = \frac{R^2}{2}\left(\sin\theta - \sin\theta\cos\theta\right)$

$= \frac{R^2}{2}\left(\sin\theta - \frac{1}{2}\sin 2\theta\right)$ NOTE: $\sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)} = \frac{\theta - \sin\theta}{\sin\theta - \frac{1}{2}\sin 2\theta} : \frac{0}{0}$ IND. FORM

(L'HOP.) $= \lim_{\theta\to 0^+} \frac{1 - \cos\theta}{\cos\theta - \cos 2\theta} : \frac{0}{0}$ IND. FORM

(L'HOP.) $= \lim_{\theta\to 0^+} \frac{\sin\theta}{-\sin\theta + 2\sin 2\theta} : \frac{0}{0}$ IND. FORM

(L'HOP.) $= \lim_{\theta\to 0^+} \frac{\cos\theta}{-\cos\theta + 4\cos 2\theta} = \frac{1}{-1+4} = \frac{1}{3}$

3. The Fresnel function

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)\,dt$$

arises in the study of the diffraction of light waves.

(a) (8 points) Evaluate $\lim_{x\to 0} \frac{S(x)}{x^3}$.

$$\lim_{x\to 0} \frac{S(x)}{x^3} = \frac{0}{0} \quad \text{IND. FORM}^{*}$$

$$(\text{L'HOP.}) = \lim_{x\to 0} \frac{\sin\left(\frac{\pi}{2}x^2\right)}{3x^2} = \frac{0}{0} \quad \text{IND. FORM}$$

$$\left(\text{FUNDAMENTAL THEOREM OF CALC.} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x) \right)$$

$$(\text{L'HOP.}) = \lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2}x^2\right)(\pi x)}{6x} = \frac{\pi}{6}$$

* NOTE THAT $S(0) = \int_0^0 \sin\left(\frac{\pi}{2}t^2\right)dt = 0$.

SINCE $S(x)$ IS DIFFERENTIABLE, IT IS CONTINUOUS.

THUS $\lim_{x\to 0} S(x) = S(0) = 0$.

(b) (8 points) Use integration by parts to show that

$$\int S(x)\,dx = xS(x) + \frac{1}{\pi}\cos\left(\frac{\pi}{2}x^2\right) + C.$$

LET $u = S(x)$ $\qquad v = x$

$du = \sin\left(\frac{\pi}{2}x^2\right)dx$ $\qquad dv = dx$

$$\int u\,dv = uv - \int v\,du = x\,S(x) - \int x \sin\left(\frac{\pi}{2}x^2\right)dx$$

LET $u = \frac{\pi}{2}x^2$ $\qquad$ THEN $\int x\sin\left(\frac{\pi}{2}x^2\right)dx = \frac{1}{\pi}\int \sin u\,du = -\frac{1}{\pi}\cos u + C$

$du = \pi x$ $\qquad = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x^2\right) + C$

$$\therefore \int S(x)\,dx = x\,S(x) + \frac{1}{\pi}\cos\left(\frac{\pi}{2}x^2\right) + C$$

4. Evaluate the following integrals.

(a) (8 points) $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta$
$dx = a\cos\theta\,d\theta$

$$\rightsquigarrow \int_0^{\pi/2} (a\sin\theta)^2\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= a^4\int_0^{\pi/2}\left(\tfrac{1}{2}\sin 2\theta\right)^2 d\theta = \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{a^4}{8}\int_0^{\pi/2} 1-\cos 4\theta\;d\theta = \frac{a^4\pi}{16}$$

(b) (8 points) $\int \frac{x^3-4x-1}{x^2-3x+2}\,dx$

$$\begin{array}{r} x + 3 \\ x^2-3x+2 \,\big)\, x^3+0x^2-4x-1 \\ -\left(x^3-3x^2+2x\right) \\ \hline 3x^2-6x-1 \\ -\left(3x^2-9x+6\right) \\ \hline 3x-7 \end{array}$$

$$\frac{x^3-4x-1}{x^2-3x+2} = x+3+\boxed{\frac{3x-7}{x^2-3x+2}}$$

PFD: $\frac{3x-7}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1}$

$3x-7 = A(x-1)+B(x-2)$

$x=2:\quad -1 = A \quad\Rightarrow\quad A=-1$

$x=1:\quad -4 = -B \quad\Rightarrow\quad B=4$

$$\therefore \int_3^4 \frac{x^3-4x+1}{x^2-3x+2}\,dx = \int_3^4 x+3-\frac{1}{x-2}+\frac{4}{x-1}\;dx = \left[\tfrac{1}{2}x^2+3x-\ln|x-2|+4\ln|x-1|\right]_3^4$$

$$= \tfrac{1}{2}(16-9)+3(4-3)+\ln 2+4\ln 3-4\ln 2 = \frac{13}{2}+4\ln 3-3\ln 2 \quad \text{or} \quad \frac{13}{2}+\ln\frac{81}{8}$$

(c) (8 points) $\int \frac{x^3+6x-2}{x^4+6x^2}\,dx$

$$x^4+6x^2 = x^2(x^2+6) \qquad \frac{x^3+6x-2}{x^2(x^2+6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+6}$$

$$x^3+6x-2 = Ax(x^2+6) + B(x^2+6) + (Cx+D)x^2$$

$$1x^3+0x^2+6x-2 = (A+C)x^3 + (B+D)x^2 + 6Ax + 6B$$

$A+C=1$ $\quad C=0$
$B+D=0$ $\quad D=\frac{1}{3}$
$6A=6$ $\quad A=1$
$6B=-2$ $\quad B=-\frac{1}{3}$

$$\int \frac{x^3+6x-2}{x^2(x^2+6)}\,dx = \int \frac{1}{x} - \frac{1/3}{x^2} + \frac{1/3}{x^2+6}\,dx$$

$$= \boxed{\ln|x| + \frac{1}{3x} + \frac{1}{3\sqrt{6}}\tan^{-1}\frac{x}{\sqrt{6}} + C}$$

(d) (8 points) $\int \sec^3 x\,dx$

$$= \int \sec x \sec^2 x\,dx$$

Let $u = \sec x$ $\qquad v = \tan x$
$du = \sec x \tan x\,dx$ $\qquad dv = \sec^2 x\,dx$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\,dx = \boxed{\frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

5. Use the table of values below to approximate $\int_0^6 f(x)\,dx$

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
$f(x)$	2	8	9	3	3	0	1	3	-6	2	-2	-6	4

(a) (8 points) using the trapezoid rule with $n = 3$

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
$f(x)$	2	8	9	3	3	0	1	3	-6	2	-2	-6	4
	x_0				x_1				x_2				x_3

$$T_3 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) \right), \quad \Delta x = \frac{6-0}{2} = 3$$

$$= \frac{3}{2}\left(2 + 2\cdot 3 + 2(-6) + 4 \right)$$

$$= \frac{3}{2}(0) = 0$$

(b) (8 points) using Simpson's rule with $n = 4$

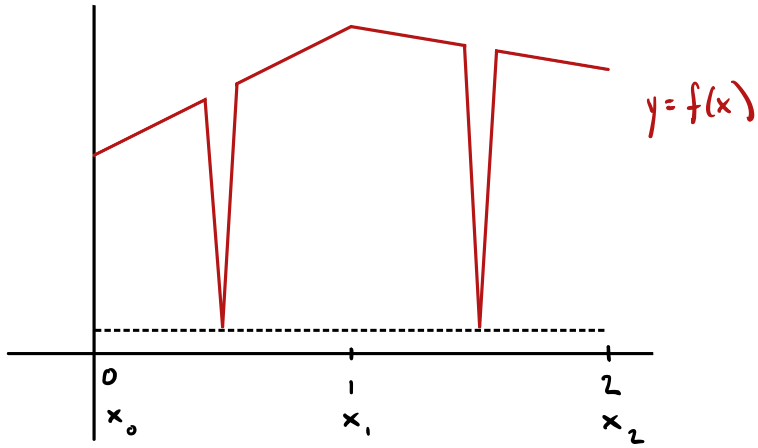
x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
$f(x)$	2	8	9	3	3	0	1	3	-6	2	-2	-6	4
	x_0			x_1			x_2			x_3			x_4

$$S_4 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right)$$

$$= \frac{1.5}{3}\left(2 + 4\cdot 3 + 2\cdot 1 + 4\cdot 2 + 4 \right)$$

$$= .5(28) = 14$$

6. (a) (8 points) Sketch the graph of a continuous function on $[0, 2]$ such that the Trapezoid rule is more accurate than the Midpoint Rule, with $n = 2$.



(b) (8 points) Sketch the graph of a continuous function on $[0, 2]$ such that the right endpoint approximation is more accurate than Simpson's Rule, with $n = 2$.