

Final Exam

Answer all 10 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. Evaluate the integrals.

(a) (6 points) $\int (x^2 - 5x)e^x dx$

INT. BY PARTS $u = x^2 - 5x$ $v = e^x$
 $du = (2x - 5) dx$ $dv = e^x dx$

$$= (x^2 - 5x)e^x - \int (2x - 5)e^x dx$$

$u = 2x - 5$ $v = e^x$
 $du = 2 dx$ $dv = e^x dx$

$$= (x^2 - 5x)e^x - \left[(2x - 5)e^x - \int 2e^x dx \right]$$

$$= (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C$$

or $(x^2 - 7x + 7)e^x + C$

(b) (6 points) $\int \sec^4 \theta \tan^2 \theta d\theta$

$\tan^2 \theta + 1 = \sec^2 \theta$

$$= \int (\tan^2 \theta + 1) \tan^2 \theta \sec^2 \theta d\theta$$

$u = \tan \theta$
 $du = \sec^2 \theta d\theta$

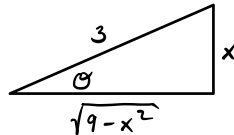
$$\leadsto \int (u^2 + 1) u^2 du = \int u^4 + u^2 du = \frac{1}{5} u^5 + \frac{1}{3} u^3 + C$$

$$\leadsto \frac{1}{5} \tan^5 \theta + \frac{1}{3} \tan^3 \theta + C$$

(c) (6 points) $\int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}} dx$

TRIG. SUBSTITUTION

Let $x = 3 \sin \theta$
 $dx = 3 \cos \theta d\theta$



Note: $\sqrt{9 - (3 \sin \theta)^2} = 3 \cos \theta$

$$= \int_0^{\pi/6} \frac{9 \sin^2 \theta}{3 \cos \theta} 3 \cos \theta d\theta = \int_0^{\pi/6} 9 \sin^2 \theta d\theta = \frac{9}{2} \int_0^{\pi/6} 1 - \cos 2\theta d\theta$$

$$= \frac{9}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/6} = \frac{9}{2} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = \frac{6\pi - 9\sqrt{3}}{8}$$

(d) (6 points) $\int \frac{1}{x(x^2+1)^2} dx$

PARTIAL FRACTIONS

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + (Bx+C)(x^3+x) + Dx^2 + Ex$$

$$= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\Rightarrow \begin{array}{rcl} A + B & & = 0 \\ & C & = 0 \\ 2A + B & + D & = 0 \\ & C & + E = 0 \\ A & & = 1 \end{array}$$

$$\Rightarrow \begin{array}{l} A = 1 \\ B = -1 \\ C = 0 \\ D = -1 \\ E = 0 \end{array}$$

$$\int \frac{1}{x(x^2+1)^2} dx = \int \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} dx \quad \begin{array}{l} u = x^2+1 \\ \frac{1}{2} du = x dx \end{array}$$

$$= \ln|x| - \frac{1}{2} \ln|x^2+1| + \frac{1}{2(x^2+1)} + C$$

(e) (6 points) $\int_0^5 \frac{1}{\sqrt[3]{5-x}} dx$

IMPROPER INTEGRAL

$$= \lim_{t \rightarrow 0^+} \int_t^5 (5-x)^{-1/3} \quad \begin{array}{l} \text{let } u = 5-x \\ -du = dx \end{array}$$

$$= \lim_{r \rightarrow 5^-} - \int_r^0 u^{-1/3} dx = \lim_{r \rightarrow 5^-} \int_0^r u^{-1/3} du = \lim_{r \rightarrow 5^-} \left. \frac{3}{2} u^{2/3} \right|_0^r$$

$$= \lim_{r \rightarrow 5^-} \frac{3}{2} r^{2/3} = \frac{3 \cdot 5^{2/3}}{2}$$

2. Consider the curve $y = x^3$, $0 \leq x \leq 4$.

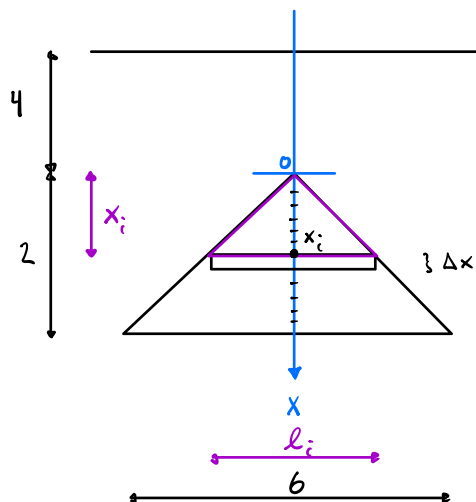
- (a) (3 points) Setup an integral for the area of the surface obtained by rotating the curve around the x -axis. Do not evaluate the integral.

$$\int_0^4 2\pi x^3 \sqrt{1 + (3x^2)^2} dx \quad \text{or} \quad \int_0^{64} 2\pi y \sqrt{1 + \left(\frac{1}{3}y^{-2/3}\right)^2} dy$$

- (b) (3 points) Setup an integral for the area of the surface obtained by rotating the curve around the y -axis. Do not evaluate the integral.

$$\int_0^4 2\pi x \sqrt{1 + (3x^2)^2} dx \quad \text{or} \quad \int_0^{64} 2\pi y^{1/3} \sqrt{1 + \left(\frac{1}{3}y^{-2/3}\right)^2} dy$$

3. (6 points) A vertical plate is submerged in water as in the figure below. Approximate the hydrostatic force against one side of the plate as a Riemann sum. Then express the force as an integral and evaluate it. Let δ equal the weight density of water per cubic foot, and leave your answer in terms of δ .



$$F_i \approx \text{AREA} \times \text{DEPTH} \times \text{DENSITY} = (l_i \Delta x)(x_i + 4) \delta$$

$$\left(\frac{l_i}{x_i} = \frac{6}{2} \Rightarrow l_i = 3x_i \right)$$

$$F_i \approx (3x_i \Delta x)(x_i + 4) \delta$$

$$F \approx \sum_{i=1}^n \delta (3x_i^2 + 12x_i) \Delta x$$

$$F = \delta \int_0^2 (3x^2 + 12x) dx = \delta \left[x^3 + 6x^2 \right]_0^2 = 32\delta$$

4. Use the table of values below to approximate $\int_0^{12} f(x) dx$ using the specified method.

- (a) (3 points) Simpson's Rule with $n = 4$.

- (b) (3 points) Midpoint Rule with $n = 3$.

You may leave your answers as a sum/product of terms.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	6	-4	7	4	6	7	-1	3	6	8	-4	5	7

(a)

	x_0		x_1		x_2		x_3		x_4				
x	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	6	-4	7	4	6	7	-1	3	6	8	-4	5	7

$$S_4 = \frac{\Delta x}{3} (f(0) + 4f(3) + 2f(6) + 4f(9) + f(12))$$

$$= \frac{2}{3} (6 + 4(4) + 2(-1) + 4(8) + 7) = 59$$

(b)

	x_0		$\bar{x}_1$		x_1		$\bar{x}_2$		x_2		$\bar{x}_3$		x_3
x	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	6	-4	7	4	6	7	-1	3	6	8	-4	5	7

$$M_3 = \Delta x (f(2) + f(6) + f(10))$$

$$= 4 (7 - 1 - 4) = 8$$

5. Consider the curve C with the following parametric equations.

$$x = t^3, \quad y = \frac{3t^2}{2}, \quad 0 \leq t \leq \sqrt{3}$$

(a) (4 points) Find the length of C .

(b) (4 points) Give an equation for the tangent line to C at the point $(1/8, 3/8)$.

$$(a) \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2} dt = \int_0^{\sqrt{3}} \sqrt{9t^4 + 9t^2} dt$$

$$= \int_0^{\sqrt{3}} 3t \sqrt{t^2 + 1} dt \quad \text{let } u = t^2 + 1$$

$$du = 2t dt \Rightarrow \frac{3}{2} du = 3t dt$$

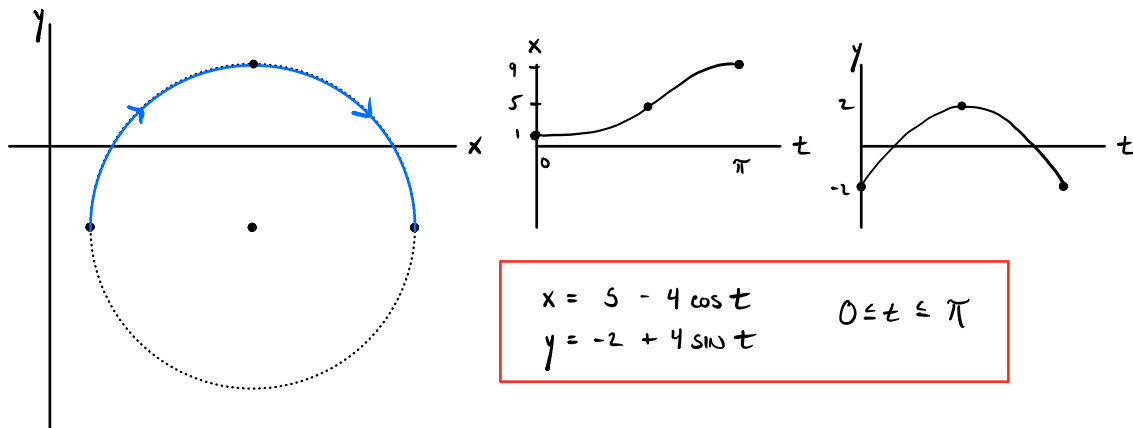
$$= \frac{3}{2} \int_1^4 \sqrt{u} du = u^{3/2} \Big|_1^4 = 8 - 1 = 7$$

$$(b) \quad x = t^3 = \frac{1}{8} \Rightarrow t = \frac{1}{2} \quad \left(y = \frac{3t^2}{2} = \frac{3}{8} \Rightarrow t = \pm \frac{1}{2} \right)$$

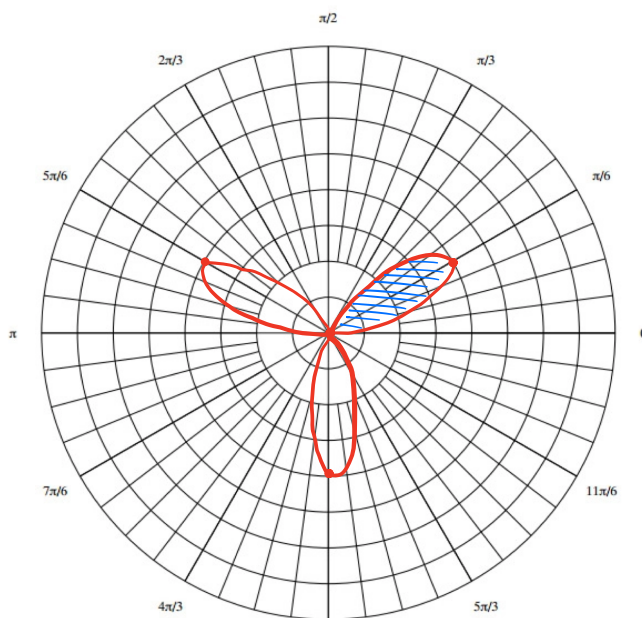
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t}{3t^2} = \frac{1}{t} \quad \xrightarrow{\text{when } t = \frac{1}{2}} = 2$$

$$y - \frac{3}{8} = 2 \left(x - \frac{1}{8} \right) \quad \text{or} \quad y = 2x + \frac{1}{8}$$

6. (4 points) Give parametric equations that describe the motion of a particle that starts at $(1, -2)$ and moves clockwise halfway around the circle with center $(5, -2)$ and radius 4. Remember to specify the domain of the parametric equations.



7. (a) (4 points) Sketch the polar curve $r = 4 \sin 3\theta$ on the polar axes below.
 (b) (6 points) Calculate the area inside one loop/petal.



$$\begin{aligned} \text{(b) } \text{Area} &= \int_0^{\pi/3} \frac{1}{2} (4 \sin 3\theta)^2 d\theta = 8 \int_0^{\pi/3} \sin^2 3\theta = 4 \int_0^{\pi/3} 1 - \cos 6\theta d\theta \\ &= 4 \left[\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \boxed{\frac{4\pi}{3}} \end{aligned}$$

8. (6 points) Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^n 5^{n-1}}$.

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \frac{(-3)^2}{2} \cdot \left(\frac{-3}{2 \cdot 5}\right)^{n-1} = \frac{9}{2} \sum_{n=1}^{\infty} \left(\frac{-3}{10}\right)^{n-1} \quad \left(\begin{array}{l} \text{GEOMETRIC SERIES} \\ \text{CONVERGES SINCE } \left| \frac{-3}{10} \right| < 1 \end{array} \right) \\
 &= \frac{9}{2} \cdot \frac{1}{1 - (-\frac{3}{10})} = \frac{90}{26} = \boxed{\frac{45}{13}}
 \end{aligned}$$

9. For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges and justify your answer. Please state the name of any test(s) for convergence/divergence that you use.

(a) (6 points) $\sum_{n=1}^{\infty} \frac{1}{n[1 + (\ln n)^2]}$

INTEGRAL TEST :

$$\int_1^{\infty} \frac{1}{x[1 + (\ln x)^2]} dx \quad \begin{array}{l} \text{let } u = \ln x \\ du = \frac{1}{x} dx \end{array}$$

$$= \int_0^{\infty} \frac{1}{1+u^2} du = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+u^2} du = \lim_{t \rightarrow \infty} \tan^{-1} u \Big|_0^t$$

$$= \lim_{t \rightarrow \infty} \tan^{-1} t - 0 = \frac{\pi}{2} \quad (\text{CONVERGES!})$$

SINCE $f(x) = \frac{1}{x[1 + (\ln x)^2]}$ IS POSITIVE & DECREASING FOR $x \geq 1$
AND $f(n) = a_n$ FOR $n = 1, 2, \dots$

AND $\int_1^{\infty} f(x) dx$ CONVERGES ,

THE SERIES CONVERGES ABSOLUTELY
BY THE INTEGRAL TEST

(b) (6 points) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+n}{n^2}$

ALTERNATING SERIES TEST: SINCE THE SERIES IS ALTERNATING & $\lim_{n \rightarrow \infty} \frac{1+n}{n^2} = 0$,

THE SERIES CONVERGES BY ALT. SERIES TEST.

DIRECT COMPARISON TEST: SINCE $\frac{1+n}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$

AND $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (HARMONIC SERIES / p -TEST)

THE SERIES DOES NOT CONVERGE ABSOLUTELY BY DIRECT COMPARISON TEST

$\therefore$ THE SERIES CONVERGES CONDITIONALLY.

10. For each of the following power series, determine the radius of convergence and the interval of convergence. Please state the name of any test(s) for convergence/divergence that you use.

(a) (6 points) $\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n+1}$

(a) Ratio Test : $\rho = \lim_{n \rightarrow \infty} \left| \frac{(2x+1)^{n+1}}{n+2} \cdot \frac{n+1}{(2x+1)^n} \right|$

$= |2x+1| \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = |2x+1|$

SERIES CONVERGES WHEN $|2x+1| = 2|x + \frac{1}{2}| < 1$

$$\left| x - \left(-\frac{1}{2}\right) \right| < \frac{1}{2}$$

$\uparrow$ $\uparrow$
 CENTER RADIUS OF CONVERGENCE

$$-\frac{1}{2} < x + \frac{1}{2} < \frac{1}{2} \Rightarrow -1 < x < 0$$

Now test endpoints: $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ CONVERGES BY ALT. SERIES TEST

$$x=0: \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{DIVERGES}$$

$\therefore$ RADIUS OF CONV. $R = \frac{1}{2}$
INTERVAL OF CONV. $[-1, 0)$

(b) (6 points) $\sum_{n=0}^{\infty} \frac{10^n x^n}{n!}$

Ratio Test: $\rho = \lim_{n \rightarrow \infty} \left| \frac{10^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n x^n} \right|$

$$= |x| \lim_{n \rightarrow \infty} \frac{10}{n+1} = 0 \quad \text{for all } x!$$

$\therefore$ RADIUS OF CONV. $R = \infty$
 INTERVAL OF CONVERGENCE $(-\infty, \infty)$