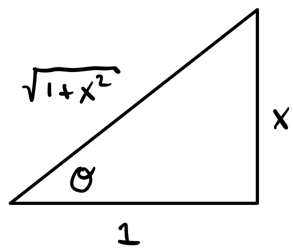
Exam 1

Answer all 5 questions for a total of 101 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. Evaluate and/or simplify each of the following expressions.
   (a) (5 points) $\sin^{-1}(\sin(\pi))$
   (b) (5 points) $\sec(\tan^{-1} x)$
   (c) (5 points) $\dfrac{d}{dt}\sinh(\ln t)$

(a) $\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = \boxed{0}$

(b) $\sec(\underbrace{\tan^{-1} x}_{\theta}) = \sec\theta = \boxed{\sqrt{1+x^2}}$



(c) $\dfrac{d}{dt}\sinh(\ln t) = \dfrac{\cosh(\ln t)}{t} = \dfrac{1}{2t}\left(e^{\ln t} + e^{-\ln t}\right)$

$= \dfrac{1}{2t}\left(t + \dfrac{1}{t}\right) = \boxed{\dfrac{1}{2}\left(1 + \dfrac{1}{t^2}\right)}$

2. Evaluate each of the following limits.
   (a) (8 points) $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x)$
   (b) (8 points) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x - \tan x}$
   (c) (8 points) $\lim\limits_{x\to\infty} x^{e^{-x}}$

(a) $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x) = \lim\limits_{u\to -\infty} \tan^{-1} u = \boxed{-\dfrac{\pi}{2}}$

(b) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x - \tan x} : \dfrac{0}{0}$

$\xrightarrow{\text{L'Ho.}} \lim\limits_{x\to 0} \dfrac{1 - \cos x}{1 - \sec^2 x} : \dfrac{0}{0}$

$$\xrightarrow{\text{L'HO.}} \lim_{x\to 0} \frac{\sin x}{-2\sec^2 x \tan x} : \frac{0}{0}$$

$$\xrightarrow{\text{L'HO.}} \lim_{x\to 0} \frac{\cos x}{-2\left(2\sec x \cdot \sec x \tan x \cdot \tan x + \sec^2 x \cdot \sec^2 x\right)}$$

$$= \lim_{x\to 0} \frac{\cos x}{-2\left(2\sec^2 x \tan^2 x + \sec^4 x\right)} = \boxed{-\frac{1}{2}}$$

(c) $\lim_{x\to\infty} x^{e^{-x}} : \infty^0$

$$\xrightarrow{\text{L'HO.}} \lim_{x\to\infty} \operatorname{EXP}\left(\ln\left(x^{e^{-x}}\right)\right)$$

$$= \operatorname{EXP}\left[\lim_{x\to\infty} \frac{\ln x}{e^x}\right] : \frac{\infty}{\infty}$$

$$\xrightarrow{\text{L'HO.}} \operatorname{EXP}\left[\lim_{x\to\infty} \frac{1}{x e^x}\right] = e^0 = \boxed{1}$$

3. Evaluate each of the following integrals.

(a) (10 points) $\int \sec^3 x \tan^5 x \, dx$

(b) (10 points) $\int \frac{5x^2 - 5x + 12}{x^3 + 4x} \, dx$

(c) (10 points) $\int x^2 e^x \, dx$

(d) (10 points) $\int_{1/4}^{\sqrt{3}/4} \sqrt{1 - 4x^2} \, dx$

(a) $\int \sec^3 x \tan^5 x \, dx = \int \sec^2 x \left(\tan^2 x\right)^2 \sec x \tan x \, dx$

$= \int \sec^2 x \left(\sec^2 x - 1\right)^2 \sec x \tan x \, dx$ let $u = \sec x$

$du = \sec x \tan x \, dx$

$\rightsquigarrow \int u^2 \left(u^2 - 1\right)^2 du = \int u^6 - 2u^4 + u^2 \, du$

$$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

$$\rightsquigarrow \boxed{\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C}$$

(b) $$\frac{5x^2 - 5x + 12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$5x^2 - 5x + 12 = A(x^2+4) + (Bx+C)x$$
$$5x^2 - 5x + 12 = (A+B)x^2 + Cx + 4A$$

$$\Rightarrow \begin{cases} A + B = 5 \\ C = -5 \\ 4A = 12 \end{cases} \quad \Rightarrow \quad A = 3, \quad B = 2, \quad C = -5$$

$$\therefore \int \frac{5x^2 - 5x + 12}{x(x^2+4)}\,dx = \int \frac{3}{x} + \frac{2x}{x^2+4} - \frac{5}{x^2+4}\,dx$$

$$= \boxed{3\ln|x| + \ln|x^2+4| - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

(c) $\int x^2 e^x\,dx$ let $u = x^2$, $v = e^x$
$du = 2x\,dx$, $dv = e^x\,dx$

$$\int u\,dv = uv - \int v\,du = x^2e^x - 2\int xe^x\,dx$$ let $u = x$, $v = e^x$
$du = dx$, $dv = e^x dx$

$$= x^2e^x - 2\left(xe^x - \int e^x\,dx\right) = x^2e^x - 2\left(xe^x - e^x\right) + C$$

$$= \boxed{e^x\left(x^2 - 2x + 2\right) + C}$$

(d) $\displaystyle\int_{1/4}^{\sqrt{3}/4} \sqrt{1-4x^2}\,dx$ Let $u = 2x$

$\frac{1}{2}du = dx$

$\rightsquigarrow \displaystyle\frac{1}{2}\int_{1/2}^{\sqrt{3}/2} \sqrt{1-u^2}\,du$ Let $u = \sin\theta$ $(\theta = \sin^{-1}u)$

$du = \cos\theta\, d\theta$

$\rightsquigarrow \displaystyle\frac{1}{2}\int_{\pi/6}^{\pi/3} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} \cos^2\theta\, d\theta$

$$= \frac{1}{4}\int_{\pi/6}^{\pi/3} 1+\cos(2\theta)\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{3}-\frac{\pi}{6}\right) + \frac{1}{2}\underbrace{\left(\sin\frac{2\pi}{3} - \sin\frac{\pi}{3}\right)}_{0}\right] = \boxed{\frac{\pi}{24}}$$

4. (10 points) Evaluate the improper integral if it converges. Otherwise, show that it diverges.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

$\displaystyle\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx$ Let $u = \ln x$ $v = -\frac{1}{x}$

$du = \frac{1}{x}dx$ $dv = \frac{1}{x^2}dx$

$$= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t + \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{-\ln t}{t} - \frac{1}{t} - (0-1) = 1 - \lim_{t\to\infty}\frac{\ln t}{t} - \underbrace{\lim_{t\to\infty}\frac{1}{t}}_{0}$$

$$= 1 - \underbrace{\lim_{t\to\infty} \frac{1/t}{1}}_{0} \quad (\text{L'Ho.}) \quad = \boxed{1}$$

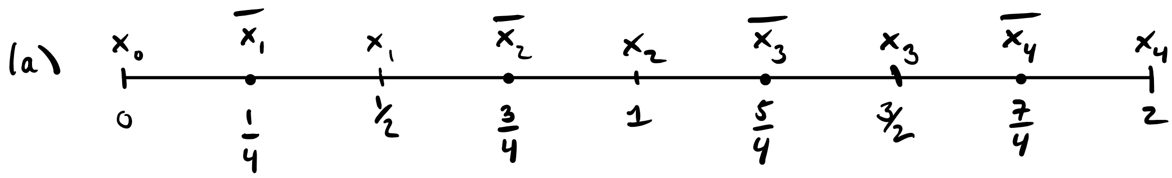
5. Approximate the definite integral

$$\int_0^2 16^x \, dx$$

(a) (6 points) using the midpoint rule with $n = 4$ subintervals.

(b) (6 points) using Simpson's rule with $n = 4$ subintervals.

(a)



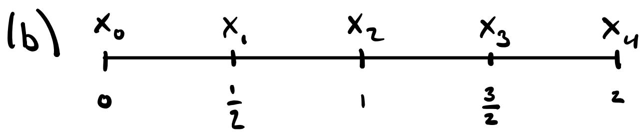
$\Delta x = \frac{1}{2}$; $x_i = x_0 + i\Delta x$; $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$; $f(x) = 16^x$

$$\int_0^2 16^x \, dx \approx \Delta x \left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4) \right)$$

$$= \frac{1}{2} \left(16^{1/4} + 16^{3/4} + 16^{5/4} + 16^{7/4} \right)$$

$$= \frac{1}{2} \left(2 + 2^3 + 2^5 + 2^7 \right) = \boxed{85}$$

(b)



$\Delta x = \frac{1}{2}$; $x_i = x_0 + i\Delta x$; $f(x) = 16^x$

$$\int_0^2 16^x \, dx \approx \frac{\Delta x}{3} \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right)$$

$$= \frac{1}{6} \left(16^0 + 4 \cdot 16^{1/2} + 2 \cdot 16 + 4 \cdot 16^{3/2} + 16^2 \right)$$

$$= \frac{1}{6} \left(1 + 16 + 32 + 256 + 256 \right) = \frac{561}{6} = \boxed{\frac{187}{2}} = 93.5$$