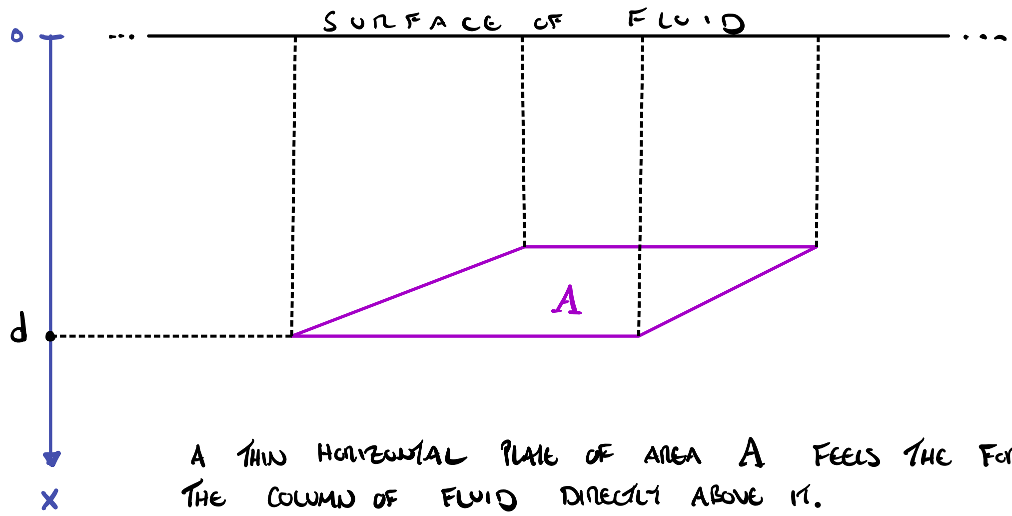
§ 8.3 Applications to Physics & Engineering

Hydrostatic Pressure & Force



A thin horizontal plate of area A feels the force of the column of fluid directly above it.

$F = mg$: mass × acceleration due to gravity

$m = V\rho$: volume × density

$V = Ad$: area × depth

$\therefore \boxed{F = \rho g A d = \delta A d}$

Metric ρg $1000\ kg/m^3 \times 9.8\ m/s^2$

Customary δ $62.5\ lbs/ft^3$

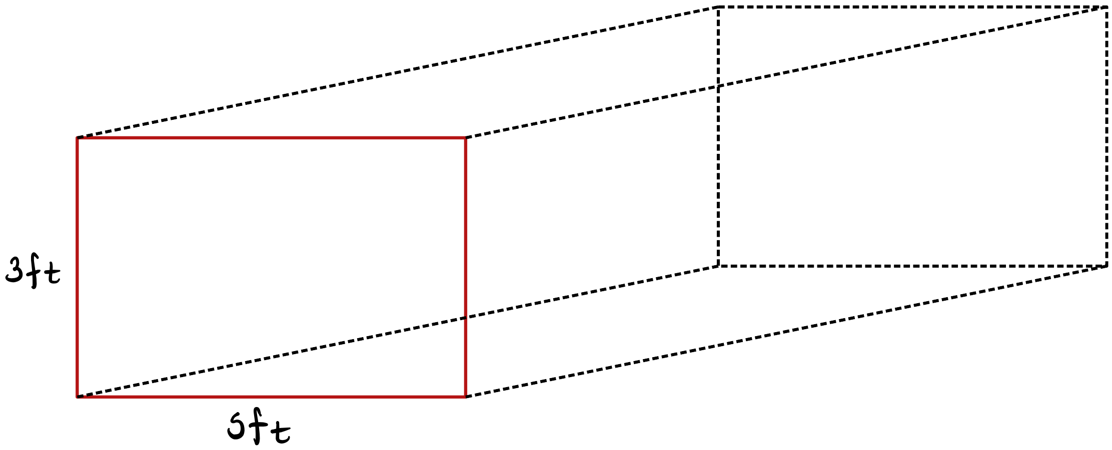
Pressure : force per unit area

$$\boxed{P = \frac{F}{A} = \rho g d = \delta d}$$

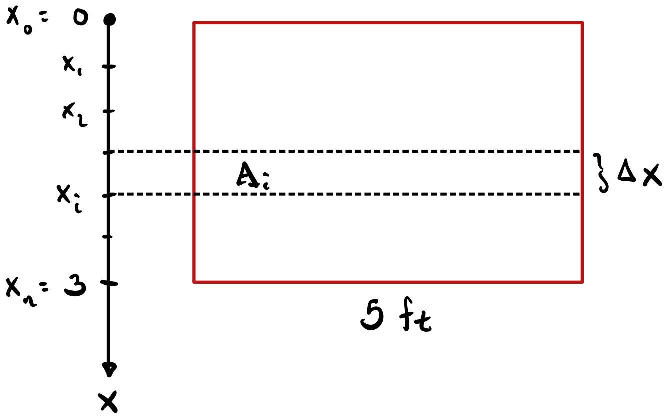
Principle of Fluid Pressure :
At any point in a fluid the pressure is the same in all directions.

$$\boxed{F = PA}$$

ex.



SUPPOSE THIS FISH TANK IS FULL OF WATER.
FIND THE HYDROSTATIC FORCE AGAINST THIS WALL OF THE TANK.



BREAK SURFACE UP INTO n HORIZONTAL STRIPS, EACH WITH WIDTH Δx

ASSUME PRESSURE AT EVERY POINT IN i^{TH} STRIP IS THE SAME, i.e. EVERY POINT IN i^{TH} STRIP IS AT SAME DEPTH x_i.

$$P_i = \delta d = 62.5\, x_i$$

THE FORCE ON THE i^{TH} STRIP IS THEN $F_i = P_i A_i = (62.5\, x_i)(5\,\Delta x)$

THE TOTAL FORCE ON SURFACE $F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} (62.5\, x_i)(5\,\Delta x)$

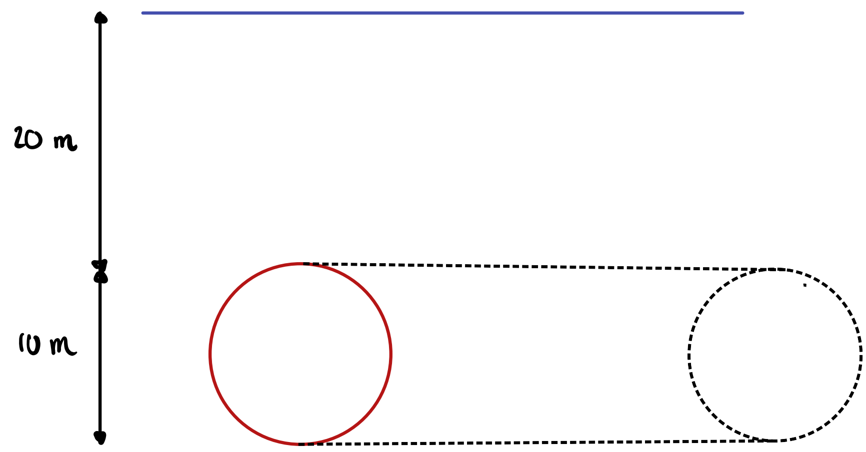
$$\approx 312.5 \sum_{i=1}^{n} x_i \Delta x \quad \text{(RIEMANN SUM)}$$

$$F = \lim_{n \to \infty} 312.5 \sum_{i=1}^{n} x_i \Delta x \quad \text{(RIEMANN INTEGRAL)}$$

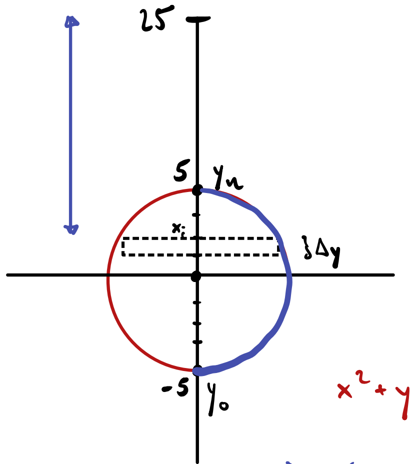
$$F = 312.5 \int_0^3 x\,dx = \frac{312.5}{2} x^2 \Big|_0^3 = 1406.25 \text{ LBS.}$$

(IF THE TANK WAS ONLY HALF-FULL: $312.5 \int_0^{1.5} x\,dx = \frac{312.5}{2} x^2 \Big|_0^{1.5} = 351.5625$ LBS.)

ex.



FIND HYDROSTATIC FORCE ON THE HIGHLIGHTED CIRCULAR WALL OF THE SUBMERGED SUBMARINE.



$$P_i = \rho g d = \rho g (25 - y_i)$$

$$F_i = P_i A_i = \rho g (25 - y_i) 2\sqrt{25 - y_i^2}\, \Delta y$$

$$x^2 + y^2 = 25$$

$$\Rightarrow x = \sqrt{25 - y^2} \Rightarrow A_i = 2\sqrt{25 - y_i^2}\, \Delta y$$

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho g (25 - x_i) 2\sqrt{25 - y_i^2}\, \Delta y$$

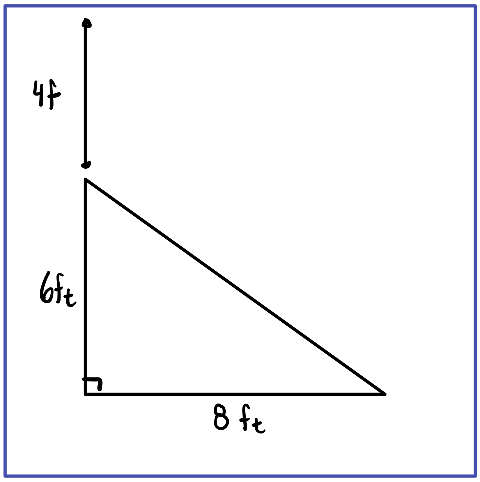
$$= 2\rho g \int_{-5}^{5} (25 - y)\sqrt{25 - y^2}\, dy$$

$$= 2(1000\ \mathrm{kg/m^3})(9.8\ \mathrm{m/s^2}) \Big[\underbrace{25 \int_{-5}^{5} \sqrt{25 - y^2}\, dy}_{\frac{1}{2}\pi \cdot 5^2} - \underbrace{\int_{-5}^{5} y\sqrt{25 - y^2}\, dy}_{\text{ODD} \Rightarrow 0}$$

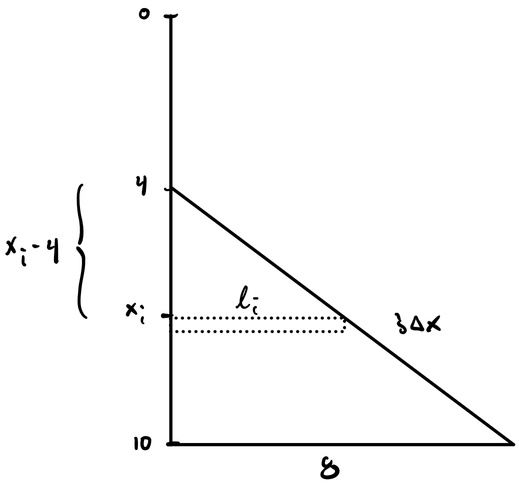
$= 19{,}242{,}255$ N (NEWTONS) $\approx$ 4.3 MILLION LBS

$\approx$ 2163 TONS

ex.



FIND THE HYDROSTATIC FORCE AGAINST ONE SIDE OF THIS TRIANGULAR PLATE SUSPENDED IN WATER.



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} P_i A_i = \lim_{n\to\infty} \sum_{i=1}^{n} (\delta x_i)\, \ell_i\, \Delta x$$

SIMILAR TRIANGLES: $\dfrac{\ell_i}{8} = \dfrac{x_i - 4}{6}$

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i \cdot \frac{4}{3}(x_i - 4)\, \Delta x$$

$$= 62.5 \cdot \frac{4}{3} \int_4^{10} x(x-4)\, dx$$

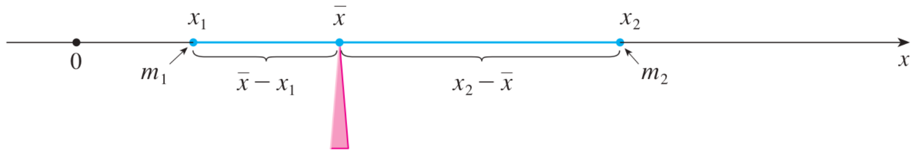
$$= \frac{250}{3} \int_4^{10} x^2 - 4x\, dx = \frac{250}{3} \left[\frac{1}{3}x^3 - 2x^2 \right]_4^{10} = \frac{250}{3} \left[\left(\frac{1000}{3} - 200 \right) - \left(\frac{64}{3} - 32 \right) \right]$$

$$= \frac{250}{3} \cdot 144 = 12{,}000 \text{ LBS.}$$

MOMENTS & CENTER OF MASS

ONE DIMENSION

LET $\bar{x}$ BE THE BALANCE POINT FOR A SYSTEM OF 2 MASSES m_1 & m_2 POSITIONED ON THE X-AXIS AT x_1 & x_2, RESPECTIVELY.



START: $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$

$(m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2 = M$ MOMENT OF THE SYSTEM WRT ORIGIN

MOMENTS OF MASSES m_1 & m_2 WRT THE ORIGIN.

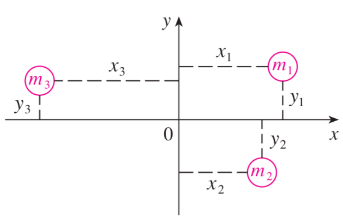
$m\bar{x} = M$

$\bar{x} = \dfrac{M}{m}$

$\bar{x} = \dfrac{M}{m} \left(= \dfrac{m_1}{m}x_1 + \dfrac{m_2}{m}x_2 \right)$ WEIGHTED AVERAGE!

WHERE m = TOTAL MASS $m_1 + m_2$

2 DIMENSIONS



MOMENT OF THE SYSTEM ABOUT THE y-AXIS:

$M_y = \sum m_i x_i$ & $\bar{x} = \dfrac{M_y}{m}$

MOMENT OF THE SYSTEM ABOUT THE X-AXIS:

$M_x = \sum m_i x_i$ & $\bar{y} = \dfrac{M_x}{m}$

$(\bar{x}, \bar{y})$ = CENTER OF MASS

NOW SUPPOSE WE HAVE A THIN PLATE (LAMINA)

WITH DENSITY ρ (UNIFORM)

IN THE SHAPE OF A REGION $\mathcal{R}$ BOUNDED BY

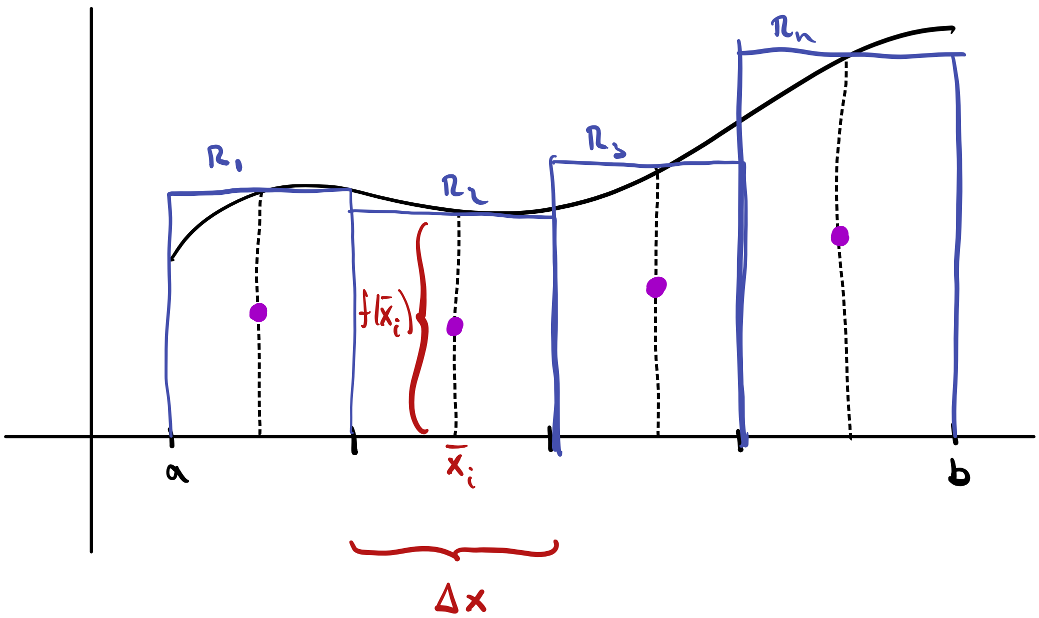
$x = a$, $x = b$, $y = 0$, $y = f(x)$.

WE APPROXIMATE $\mathcal{R}$ WITH n RECTANGLES $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$

EACH WITH WIDTH $\Delta x = \dfrac{b-a}{n}$ AND HEIGHT $f(\bar{x}_i)$,

WHERE $\bar{x}_i$ = MIDPOINT OF i^{TH} SUBINTERVAL.

THEN THE CENTER OF MASS OF $\mathcal{R}_i$ IS $\left(\bar{x}_i, \tfrac{1}{2} f(\bar{x}_i)\right)$.



THEN THE MOMENT OF $\mathcal{R}_i$ ABOUT y-AXIS IS

$$\begin{aligned}(\text{MASS OF } \mathcal{R}_i)\,\bar{x}_i &= (\text{DENSITY} \times \text{AREA})\,\bar{x}_i \\ &= [\rho f(\bar{x}_i)\Delta x]\,\bar{x}_i = \rho \bar{x}_i f(\bar{x}_i)\Delta x\end{aligned}$$

$\therefore$ MOMENT OF $\mathcal{R}$ ABOUT y-AXIS IS

$$M_y \approx \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x$$

AND $\boxed{M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x)\,dx}$

SIMILARLY, THE MOMENT OF R_i ABOUT X-AXIS IS

$$\left(\text{MASS OF } R_i\right)\frac{1}{2}f(\bar{x}_i) = \left(\text{DENSITY} \times \text{AREA}\right)\frac{1}{2}f(\bar{x}_i)$$

$$= \left[\rho f(\bar{x}_i)\Delta x\right]\frac{1}{2}f(\bar{x}_i) = \rho\,\frac{1}{2}f(\bar{x}_i)^2\Delta x$$

$\therefore$ MOMENT OF R ABOUT X-AXIS IS

$$M_x \approx \sum_{i=1}^{n} \rho\,\frac{1}{2}f(\bar{x}_i)^2\Delta x$$

AND $$\boxed{M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\rho\,\frac{1}{2}f(\bar{x}_i)^2\Delta x = \rho\int_a^b \frac{1}{2}f(x)^2\,dx}$$

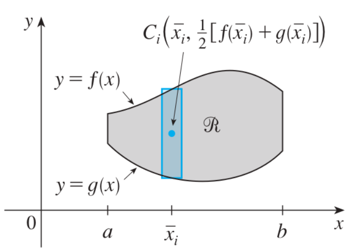
AND $$\bar{x} = \frac{M_y}{M} = \frac{\cancel{\rho}\int_a^b x f(x)\,dx}{\cancel{\rho}\int_a^b f(x)\,dx} = \frac{1}{A}\int_a^b x f(x)\,dx$$

$$\bar{y} = \frac{M_x}{M} = \frac{\cancel{\rho}\int_a^b \frac{1}{2}f(x)^2\,dx}{\cancel{\rho}\int_a^b f(x)\,dx} = \frac{1}{A}\int_a^b \frac{1}{2}f(x)^2\,dx$$

$(\bar{x},\bar{y})$ IS CALLED THE CENTROID OF THE REGION R

a.k.a. CENTER OF MASS.

MORE GENERALLY,



as illustrated in Figure 13, then the same sort of argument that led to Formula used to show that the centroid of $\mathscr{R}$ is $(\bar{x}, \bar{y})$, where

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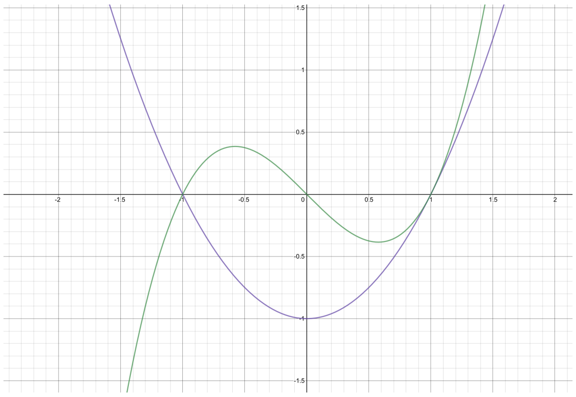
$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx$$



37. Find the centroid of the region bounded by the curves $y = x^3 - x$ and $y = x^2 - 1$. Sketch the region and plot the centroid to see if your answer is reasonable.

$$A = \int_{-1}^{1} \left[\overbrace{(x^3 - x)}^{0} - (x^2 - 1) \right] dx = x - \frac{1}{3}x^3 \Big|_{-1}^{1} = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

$$\bar{x} = \frac{1}{A} \int_{-1}^{1} x \left[(x^3 - x) - (x^2 - 1) \right] dx = \frac{3}{4} \cdot 2 \left[\frac{1}{5}x^5 - \frac{1}{3}x^3 \right]_0^1 = \frac{3}{2}\left(-\frac{2}{15}\right) = \frac{-1}{5}$$

$$\bar{y} = \frac{1}{A} \int_{-1}^{1} \frac{1}{2} \left[(x^3 - x)^2 - (x^2 - 1)^2 \right] dx = \frac{3}{4}\left[-\frac{16}{35} \right] = \frac{-12}{35}$$

$$x^6 - 2x^4 + x^2 - (x^4 - 2x^2 + 1)$$

$$x^6 - 3x^4 + 3x^2 - 1 = (x^2 - 1)^3$$

$$\left[\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x \right]_0^1 = \frac{5-21}{35} = \frac{-16}{35}$$