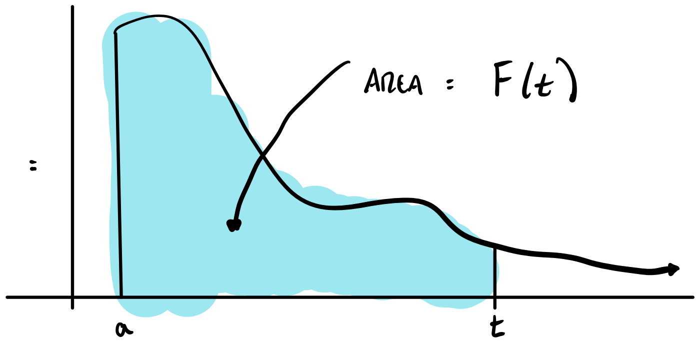
§ 8.8 Improper Integrals

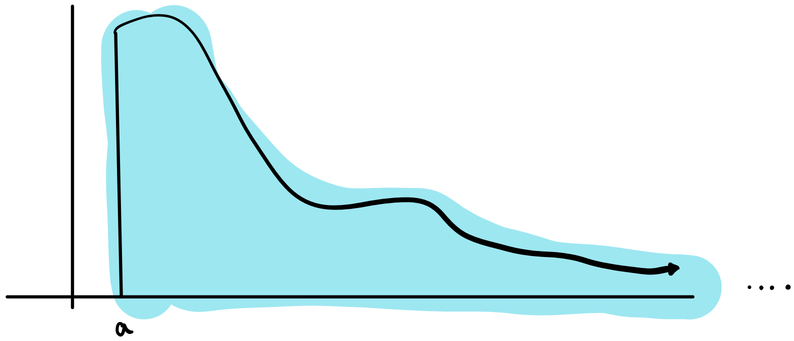
Let $F(t) = \int_a^t f(x)\,dx$ =



F.T.C.

$\left[\; F'(t) = f(t) \;\right]$

$\lim_{t\to\infty} F(t) = ?$



$$\int_a^\infty f(x)\,dx \overset{\text{Def}}{:=} \lim_{t\to\infty} \int_a^t f(x)\,dx = \lim_{t\to\infty} F(t)$$

Improper integral Type I

∞ interval of integration

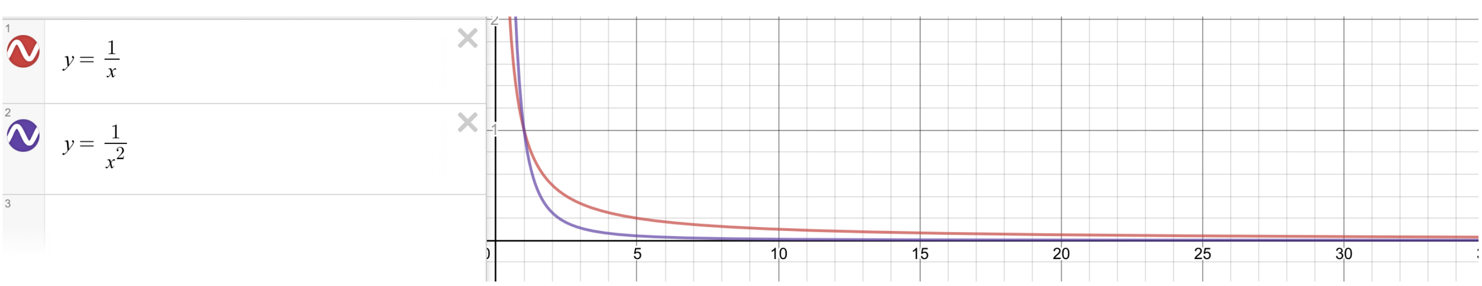
Improper integral of **Type I**: integrating over an **infinite interval.**

ex. $\int_1^\infty \frac{1}{x^2}\,dx$ converges (limit exists)

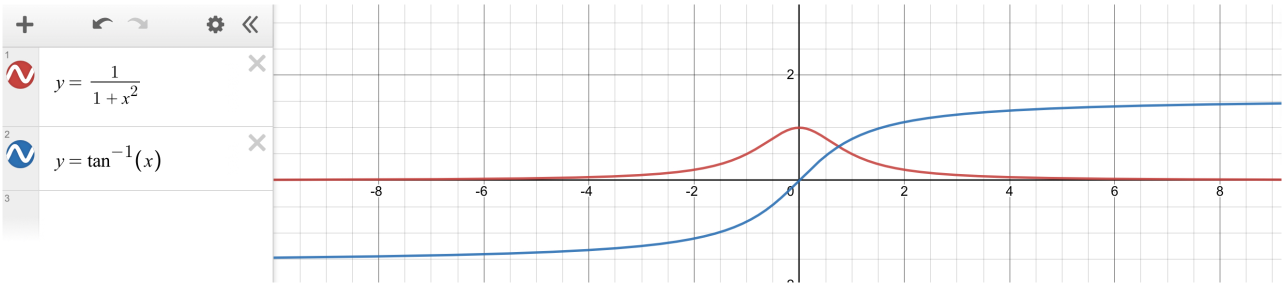
ex. $\int_1^\infty \frac{1}{x}\,dx$ diverges (limit does not exist)

Procedure:

1. Use definition
2. Evaluate integral
3. Take limit



ex. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$



DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Question: For which values of p does $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ converge? diverge?

Note: $$\lim_{t\to\infty} t^n = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \begin{cases} \displaystyle\lim_{t\to\infty} \frac{1}{1-p} x^{1-p} \Big|_1^t & \text{if } p \neq 1 \\ \displaystyle\lim_{t\to\infty} \ln|x| \Big|_1^t & \text{if } p = 1 \end{cases}$$

$$= \begin{cases} \displaystyle\lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p} - 1\right) = \begin{cases} \dfrac{1}{p-1} & \text{if } 1-p < 0 \ (p > 1) \\ \infty & \text{if } 1-p > 0 \ (p < 1) \end{cases} \\ \displaystyle\lim_{t\to\infty} \ln|t| - 0 = \infty \end{cases}$$

(converges) for $p > 1$

(diverges) for $p < 1$

(diverges) for $p = 1$

"p-rule"

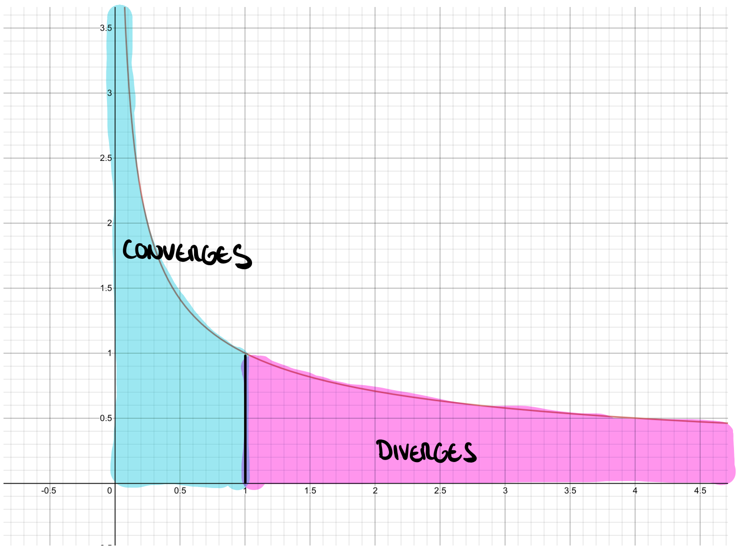
$\boxed{2}$ $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

e.g. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges $(p = 2 > 1)$ ✓

$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges $(p = 1 \leq 1)$ ✓

ex. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (p-rule with $p = \frac{1}{2} \leq 1$)

Question: What about $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$?



- $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$

improper integral type II

(infinite discontinuity)

$\left(\ldots = \displaystyle\lim_{t\to 0^+} \left[2 - 2\sqrt{t}\right] = 2 \right)$

- $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx$

improper integral type I

(infinite interval)

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

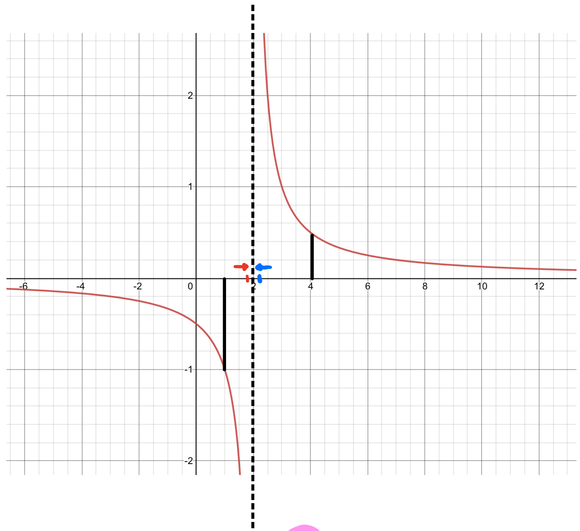
The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

WATCH OUT! **ex.**

$$\int_1^4 \frac{1}{x-2}\,dx \quad \text{(WRONG)} = \ln|x-2|\Big|_1^4 = \ln 2 - \ln|-1| = \ln 2$$



YOU CANNOT INTEGRATE OVER DISCONTINUITIES!

BREAK INT. UP INTO 2 IMP. INT. OF TYPE II!

PUT POINT OF DISCONTINUITY AS BOUND OF INTEGRATION.

$$\int_1^4 \frac{1}{x-2}\,dx = \int_1^{2} \frac{1}{x-2}\,dx + \int_{2}^4 \frac{1}{x-2}\,dx$$

$$= \lim_{t\to 2^-} \int_1^t \frac{1}{x-2}\,dx + \lim_{t\to 2^+} \int_t^4 \frac{1}{x-2}\,dx$$

$$= \underbrace{\lim_{t\to 2^-} \Big[\ln|t-2| - \ln|1-2|\Big]}_{\substack{\text{LIMIT. D.N.E} \\ \text{(DIVERGES)}}} + \underbrace{\lim_{t\to 2^+} \Big[\ln|4-2| - \ln|t-2|\Big]}_{\substack{\text{LIMIT D.N.E.} \\ \text{(DIVERGES)}}} = \text{DNE}$$

NOTE: $\infty - \infty \neq 0$

$1 - 1 + 1 - 1 + \ldots = ?$

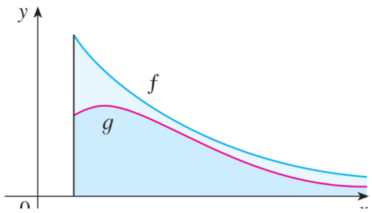
$(1-1) + (1-1) + (1-1) + \ldots = 0$

$1 + (-1+1) + (-1+1) + (-1+1) + \ldots = 1$

$(1+1+1+\ldots) + (-1-1-1-1-\ldots) = \infty - \infty\ ?$

INFINITE SUMS ONLY CONVERGE WHEN THEY DO SO REGARDLESS OF ORDER & GROUPING.

Comparison Thm



Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. ■

49–54 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

50. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

52. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

53. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

54. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$