# §7.4 PARTIAL FRACTIONS

A TECHNIQUE FOR INTEGRATING ANY PROPER RATIONAL FUNCTION

$\frac{P(x)}{Q(x)}$ WHERE $P$ & $Q$ ARE POLYNOMIALS & $\text{DEG}(P) < \text{DEG}(Q)$.

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**STEP 0:** IF IMPROPER → POLY. LONG DIVISION (OR EQUIVALENT)

e.g. $$\int \frac{x^4+1}{x^2+1}\,dx = \int x^2 - 1 + \frac{2}{x^2+1}\,dx$$
$$= \frac{1}{3}x^3 - x + 2\tan^{-1}x + C$$

(Note: $\frac{x^4+1}{x^2+1} = \frac{x^4-1+2}{x^2+1} = \frac{(x^2+1)(x^2-1)+2}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}$ )

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IDEA Since $\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x^2+3x+2}$,

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{x+1} - \frac{1}{x+2} = \ln|x+1| - \ln|x+2| + C$$

THE METHOD OF PARTIAL FRACTIONS IS AN ALGORITHM FOR DECOMPOSING PROPER RATIONAL FUNCTIONS INTO SUMS OF PARTIAL FRACTIONS.

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**STEP 1.** FACTOR DENOMINATOR $Q$ COMPLETELY.

FUNDAMENTAL THEOREM OF ALGEBRA

EVERY POLYNOMIAL CAN BE FACTORED INTO LINEAR FACTORS $ax+b$

& IRREDUCIBLE QUADRATIC FACTORS $ax^2+bx+c$, $b^2-4ac<0$

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**STEP 2.** WRITE DOWN FORM/TEMPLATE OF PFD (PARTIAL FRACTION DECOMPOSITION)

THERE ARE RULES TO MEMORIZE!

| | FACTOR | TERM(S) IN PFD |
|---|---|---|
| DIST. LIN. | $(ax+b)$ | $\dfrac{A}{ax+b}$ |
| REPEAT LIN. | $(ax+b)^n$ | $\dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2} + \dots + \dfrac{N}{(ax+b)^n}$ |
| DIST. QUAD. | $(ax^2+bx+c)$ | $\dfrac{Ax+B}{ax^2+bx+c}$ |
| REPEAT QUAD. | $(ax^2+bx+c)^n$ | $\dfrac{Ax+B}{ax^2+bx+c} + \dfrac{Cx+D}{(ax^2+bx+c)^2} + \dots + \dfrac{Mx+N}{(ax^2+bx+c)^n}$ |

**ex.** GIVE THE FORM OF THE PFD:

(a) $\dfrac{3x}{(x-1)(x-2)}$ (b) $\dfrac{3x+1}{(x-1)^2(x-2)}$

(c) $\dfrac{3x}{(x-1)(x^2+1)}$ (d) $\dfrac{x+3}{(x-1)(x^2+1)^2}$

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**STEP 3.** SOLVE FOR UNDETERMINED COEFFICIENTS $A, B, C, \dots$

1. Worked Example:

$$\frac{6x^2-6x-6}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

Clear fractions:

$$6x^2-6x-6 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$$

And this statement must be true for all $x$. In particular, it must be true for $x=1, x=-2$ and $x=3$ (we chose these to zero out the others). Substituting, we get

$$A=1 \qquad B=2 \qquad C=3$$

2. Worked Example:

$$\frac{x^2-2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

Clear fractions. In this case, it might be best to solve for the coefficients in a slightly different manner- Equate the coefficients to the polynomials on the left and right:

$$x^2-2 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + Cx + 2A$$

so that:

$$1 = A+B, 0 = C, 2A = -2$$

so: $A=-1, B=2$ and $C=0$:

$$\frac{x^2-2}{x(x^2+2)} = \frac{-1}{x} + \frac{2x}{x^2+2}$$

EXAMPLES

1. $$\frac{x^2+1}{x^2+3x+2} = 1 - \frac{5}{x+2} + \frac{2}{x+1}$$ (IMPROPER, PLUG IN VAL.)

* 2. $$\frac{2x+3}{(x+1)^2} = \frac{2}{x+1} + \frac{1}{(x+1)^2}$$ (PLUG IN +)

3. $$\frac{4x^2-7x-12}{x(x+2)(x-3)} = \frac{2}{x} + \frac{9}{5}\frac{1}{x+2} + \frac{1}{5}\frac{1}{x-3}$$ (PLUG IN)

4. $$\frac{x^2+3}{x^3+2x} = \frac{3}{2}\frac{1}{x} - \frac{1}{2}\frac{x}{x^2+2}$$ (EQUAL COEFF.)

* 5. $$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1}$$ (EQUAL COEFF.)

6. $$\frac{3x^3-x+12}{x^2-1} = 3x + \frac{7}{x-1} - \frac{5}{x+1}$$ (IMPROPER, PLUG IN)