§7.3 Trig Substitution

Motivation

Consider $\displaystyle\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \xleftrightarrow[\text{"trig sub"}]{\text{"u-sub"}} \int_0^1 \sqrt{1-x^2}\,dx$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$

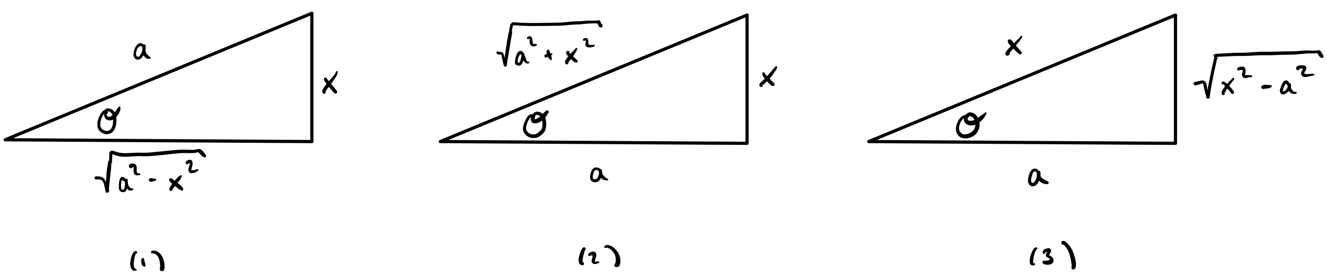
In fact, trig identities make trig integral easier!

$$\int_0^{\pi/2} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta = \int_0^{\pi/2} \cos^2\theta\,d\theta \qquad \left(0 \le \theta \le \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} 1+\cos 2\theta\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/2} = \frac{\pi}{4}$$

Table of Trigonometric Substitutions

	Expression	Substitution	Identity
(1)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
(2)	$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
(3)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$



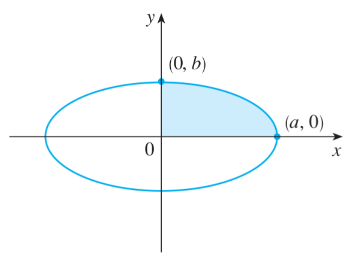
---

ex. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{1}{x^2\sqrt{9x^2-1}}\,dx \rightsquigarrow \frac{1}{3}\int_{\sqrt{2}}^{2} \frac{3^2}{u^2\sqrt{u^2-1}}\,du \rightsquigarrow 3\int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta\,\tan\theta}\,\sec\theta\tan\theta\,d\theta$

EXAMPLE 3 Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



24. $\int_0^1 \sqrt{x - x^2}\, dx$

HINT: $u = \sqrt{x}$

18. $\int \frac{dx}{[(ax)^2 - b^2]^{3/2}}$

$u = ax$

$\frac{1}{a}\, du = dx$

$\rightsquigarrow \frac{1}{a} \int \frac{1}{\sqrt{u^2 - b^2}^{\,3}}\, du$

$u = b \sec \theta$

$du = b \sec \theta \tan \theta \, d\theta$

$\rightsquigarrow \frac{1}{a} \int \frac{b \sec \theta \tan \theta}{\tan^3 \theta}\, d\theta = \frac{b}{a} \int \frac{\cos \theta}{\sin^2 \theta}\, d\theta$

20. $\int \frac{x}{\sqrt{1 + x^2}}\, dx$

DOES NOT REQUIRE TRIG SUB! U-SUB IS EASIER!

(BUT TRIG SUB WOULD WORK)