§6.7 Hyperbolic Functions

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

23. Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$

(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$

(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$

(g) $\lim_{x\to0^+} \coth x$ (h) $\lim_{x\to0^-} \coth x$

(i) $\lim_{x\to-\infty} \operatorname{csch} x$ (j) $\lim_{x\to\infty} \dfrac{\sinh x}{e^x}$

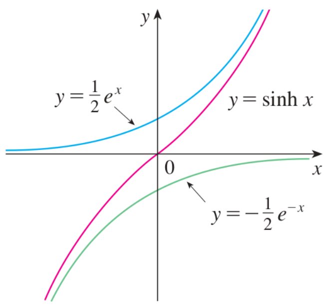

FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

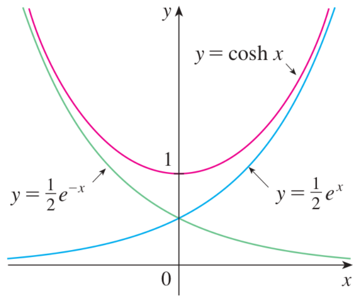

FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

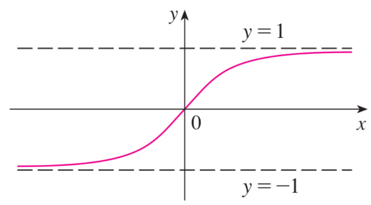

FIGURE 3
$y = \tanh x$

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

ALL OF THESE FOLLOW DIRECTLY FROM THE DEFINITIONS ABOVE

(MORE IDENTITIES EXIST! YOU DON'T NEED TO KNOW THEM.)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

GIVEN ANY $t \in \mathbb{R}$, THE POINT $(\cosh t, \sinh t)$ LIES ON THE HYPERBOLA $x^2 - y^2 = 1$

$$\frac{1}{4}(e^x - e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x + e^{-x})(e^y - e^{-y})$$

$$= \frac{1}{4}\left[e^{x+y} + \cancel{e^{x-y}} - \cancel{e^{-x+y}} - e^{-x-y} + e^{x+y} - \cancel{e^{x-y}} + \cancel{e^{-x+y}} - e^{-x-y} \right]$$

$$= \frac{1}{4}\left[2e^{x+y} - 2e^{-x-y} \right] = \frac{1}{2}\left(e^{x+y} - e^{-(x+y)} \right) = \sinh(x+y) \checkmark$$

ALL OF THESE FOLLOW DIRECTLY FROM THE DEFINITIONS ABOVE

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

30–45 Find the derivative. Simplify where possible.

30. $f(x) = e^x \cosh x$

31. $f(x) = \tanh \sqrt{x}$

32. $g(x) = \sinh^2 x$

33. $h(x) = \sinh(x^2)$

34. $F(t) = \ln(\sinh t)$

35. $G(t) = \sinh(\ln t)$

36. $y = \operatorname{sech} x\,(1 + \ln \operatorname{sech} x)$

37. $y = e^{\cosh 3x}$

38. $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$

61. $\displaystyle\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

62. $\displaystyle\int \tanh x\,dx$

63. $\displaystyle\int \frac{\cosh x}{\cosh^2 x - 1}\,dx$

64. $\displaystyle\int \frac{\operatorname{sech}^2 x}{2 + \tanh x}\,dx$

INVERSE HYPERBOLIC FUNCTIONS

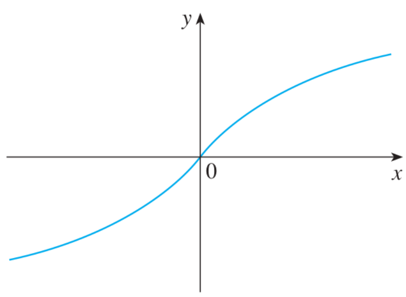

FIGURE 8 $y = \sinh^{-1} x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

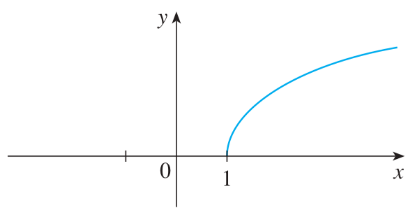

FIGURE 9 $y = \cosh^{-1} x$
domain $= [1, \infty)$ range $= [0, \infty)$

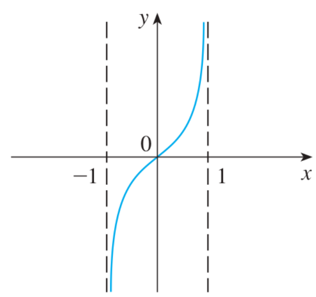

FIGURE 10 $y = \tanh^{-1} x$
domain $= (-1, 1)$ range $= \mathbb{R}$

DEFINITIONS:

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

$$y = \tanh^{-1} x \iff \tanh y = x$$

ALTERNATIVE DEFINITIONS:

3 * $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$

4 $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$

5 $\tanh^{-1} x = \frac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right) \qquad -1 < x < 1$

* LET $y = \sinh^{-1} x$.

$\sinh y = x \Rightarrow x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow 2x = e^y - e^{-y}$

$\Rightarrow 0 = \cancel{e^{-y}}(e^{2y} - 2xe^y - 1) \quad (e^{-y} \neq 0)$

LET $w = e^y$: $0 = w^2 - 2xw - 1$ (QUADRATIC IN w)

$a = 1, \; b = -2x, \; c = -1$

$\Rightarrow w = \dfrac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$

$w > 0 \Rightarrow w = x + \sqrt{x^2 + 1} \Rightarrow e^y = x + \sqrt{x^2 + 1}$

$\Rightarrow \sinh^{-1} x = y = \ln\left(x + \sqrt{x^2 + 1}\right)$ ✓

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$$

e.g. LET $y = \sinh^{-1} x$. FIND y'.

$$\sinh y = x \xrightarrow{\text{IMP. DIFF.}} \cosh y \cdot y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\pm\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}} \quad \checkmark$$

$y' > 0$ $\cosh y > 0$

65. $\displaystyle\int_4^6 \frac{1}{\sqrt{t^2-9}}\,dt$

66. $\displaystyle\int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt$

67. $\displaystyle\int \frac{e^x}{1-e^{2x}}\,dx$