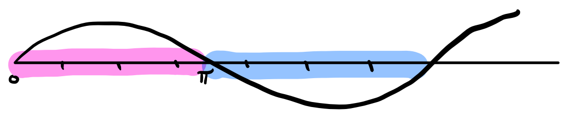
§11.6 ABSOLUTE CONVERGENCE ; RATIO , ROOT TESTS

ABSOLUTE CONVERGENCE TEST

IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES ,

i.e., ABSOLUTE CONVERGENCE $\Rightarrow$ CONVERGENCE .

ex. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n\sqrt{n}}$



$$= \frac{\sin(1)}{1\sqrt{1}} + \frac{\sin(2)}{2\sqrt{2}} + \frac{\sin(3)}{3\sqrt{3}} + \frac{\sin(4)}{4\sqrt{4}} + \frac{\sin(5)}{5\sqrt{5}}$$

NOT TRUE

$$\underbrace{0 \le \frac{\sin(n)}{n\sqrt{n}}}_{\text{NOT TRUE}} \le \frac{1}{n\sqrt{n}}$$

WE CANNOT USE D.C.T.

INSTEAD , CONSIDER $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n\sqrt{n}} \right|$

$$0 \le \left| \frac{\sin(n)}{n\sqrt{n}} \right| \le \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

BIGGER TERMS !

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ CONVERGES (p-SERIES TEST : $p = \frac{3}{2} > 1$ ✓)

STILL CONVERGES.

$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n\sqrt{n}} \right|$ CONVERGES BY D.C.T.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin(n)}{n\sqrt{n}}$ ALSO CONVERGES BY ABS. CONV. TEST.

Def: IF $\sum_{n=1}^{\infty} a_n$ CONVERGES, BUT $\sum_{n=1}^{\infty} |a_n|$ DIVERGES,

THEN WE SAY THE SERIES $\sum_{n=1}^{\infty} a_n$ CONVERGES CONDITIONALLY.

THAT IS, A SERIES THAT CONVERGES BUT DOES NOT CONVERGE ABSOLUTELY CONVERGES CONDITIONALLY.

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ex. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ CONVERGES (ALT SERIES TEST: $(-1)^{n+1}\frac{1}{n} \to 0$)

BUT IT DOES NOT CONVERGE ABSOLUTELY ($\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES)

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ CONVERGES CONDITIONALLY.

Two Tests For Absolute Convergence: 1. Ratio Test 2. Root Test

THE RATIO TEST:

LET $\sum a_n$ BE ANY SERIES & SUPPOSE

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

THEN $\sum a_n \begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1 \end{cases}$

ρ GREEK LOWERCASE r "RATIO"

PROOF: FOR LARGE n, SAY $n \geq N$, WE HAVE

$$\left| \frac{a_{n+1}}{a_n} \right| \approx \rho \implies |a_{n+1}| \approx \rho |a_n|$$

$$a_{1000} \xrightarrow{\times\rho\,\approx} a_{1001} \xrightarrow{\times\rho\,\approx} a_{1002} \xrightarrow{\times\rho\,\approx} a_{1003}$$

SERIES $\approx$ GEOMETRIC !

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \underbrace{\sum_{n=N}^{\infty} |a_n|}$$

$$\approx |a_N| + \rho|a_N| + \rho^2|a_N| + \ldots$$

$$\approx \sum_{n=1}^{N-1} |a_n| + \underbrace{\sum_{n=1}^{\infty} |a_N| \rho^{n-1}}$$

CONVERGES IF $\rho < 1$

DIVERGES IF $\rho > 1$

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES, AND $\displaystyle\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)^2}{1/n^2}\right|$

$$= \lim_{n\to\infty} \left|\frac{n^2}{n^2+2n+1}\right| = 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES, AND $\displaystyle\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)}{1/n}\right|$

$$= \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

$\blacksquare$

ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$ CONV. OR DIV.?

IF IT CONVERGES, DOES IT CONVERGE ABSOLUTELY, OR CONDITIONALLY?

SINCE THE TERMS ALTERNATE POS/NEG &

$\frac{n+2}{3^n}$ IS DECREASING AND $\lim_{n\to\infty} \frac{n+2}{3^n} = 0$ (L'HO)

THE ALTERNATING SERIES TEST $\Rightarrow$ SERIES CONVERGES.

CONVERGE ABSOLUTELY?

RATIO TEST: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)+2}{3^{n+1}}}{\frac{n+2}{3^n}} \right|$

$$= \lim_{n\to\infty} \left[\frac{n+3}{\cancel{3^{n+1}}} \cdot \frac{\cancel{3^n}}{n+2} \right] = \lim_{n\to\infty} \underbrace{\left(\frac{n+3}{n+2}\right)}_{1}\left(\frac{1}{3}\right) = \frac{1}{3}$$

YES, $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$ CONVERGES ABSOLUTELY (RATIO TEST $\rho = \frac{1}{3} < 1$ ✓)

ex. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 (n+2)!}{n! \, 3^{2n}}$

NOTE RATIO TEST IS GREAT FOR SERIES WITH TERMS INVOLVING FACTORIALS.

RATIO TEST: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \frac{(n+1)^2 ((n+1)+2)!}{(n+1)! \, 3^{2(n+1)}} \cdot \frac{n! \, 3^{2n}}{n^2 (n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \left(\frac{3^{2n}}{3^{2n+2}}\right)\left(\frac{(n+3)!}{(n+2)!}\right)\left(\frac{n!}{(n+1)!}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{9}\right) \left(\frac{\boxed{(n+3)}\cancel{(n+2)}\cancel{(n+1)}\cancel{n}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{(n+2)}\cancel{(n+1)}\cancel{n}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}} \right) \left(\frac{\cancel{n!}}{\boxed{(n+1)}\cancel{n!}} \right)$$

$$= \frac{1}{9} \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^2}_{1} \underbrace{\left(\frac{n+3}{n+1}\right)}_{1} = 1/9$$

$\rho = 1/9 \Rightarrow$ SERIES CONVERGES ABSOLUTELY! (STRONGER RESULT)

[THE SERIES CONVERGES BY ALT. SERIES TEST. (WEAKER RESULT)

$$\lim_{n\to\infty} \frac{n^2 (n+2)!}{n!\, 3^{2n}} = \lim_{n\to\infty} \frac{n^2 (n+2)(n+1)\cancel{n!}}{\cancel{n!}(3^2)^n}$$

$$= \lim_{n\to\infty} \frac{n^2(n+2)(n+1)}{9^n} : \frac{\infty}{\infty} \longrightarrow \lim_{n\to\infty} \frac{n^2(n+2)(n+1)}{9^n} = 0$$
]

THE ROOT TEST

LET $\sum a_n$ BE ANY SERIES & SUPPOSE THAT

$$\rho = \lim_{n\to\infty} |a_n|^{1/n}.$$

THEN $\sum a_n$ $\begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1 \end{cases}$

PROOF: FOR n LARGE ENOUGH, SAY $n \geq N$, WE HAVE

$$|a_n|^{1/n} \approx \rho \Rightarrow |a_n| \approx \rho^n$$

$$\therefore \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$$

$$\approx \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} \rho^n$$

GEOMETRIC SERIES!

CONVERGES IF $\rho < 1$ $\quad$ (1^{∞} IND. FORM)

DIVERGES IF $\rho > 1$ $\quad$ ($\rho = 1$ INCONCL.)

NOTE: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES, $\displaystyle\rho = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{2/n}} = \lim_{n\to\infty} n^{-2/n}$

$$= \text{Exp} \lim_{n\to\infty} \left[\frac{-2\ln(n)}{n}\right] = \text{Exp}(0) = 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES, $\displaystyle\rho = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \ldots = 1$

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$ $\quad$ ROOT TEST: $\displaystyle\rho = \lim_{n\to\infty} |a_n|^{1/n}$

$$= \lim_{n\to\infty} \left[\frac{4^n}{(3n)^n}\right]^{1/n} = \lim_{n\to\infty} \frac{4}{3n} = 0$$

$\therefore$ ROOT TEST $\Rightarrow$ SERIES CONVERGES ABSOLUTELY.

ex. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$ $\quad$ ROOT TEST: $\displaystyle\rho = \lim_{n\to\infty} \left[\left(\frac{4n+3}{3n-5}\right)^n\right]^{1/n}$

$$= \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1$$

$\therefore$ SERIES DOES NOT CONVERGE ABSOLUTELY.

$$\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n = -\frac{7}{2} + \underbrace{\sum_{n=2}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n}$$

DOES NOT CONVERGE ABSOLUTELY. BUT ALL TERMS POSITIVE ANYWAY, SO IT DOES NOT CONVERGE.

$\therefore$ SERIES DIVERGES.