§ 11.5 ALTERNATING SERIES

Def: A SERIES $\sum_{n=1}^{\infty} a_n$ IS ALTERNATING IF IT HAS THE FORM

$$\boxed{\pm \sum_{n=1}^{\infty} (-1)^{n+1} u_n \,, \quad u_n > 0 .}$$

CAN BE WRITTEN THIS WAY.

VALUE , $u_n = |a_n|$

THIS IS THE ALTERNATING FACTOR $1, -1, 1, -1, \ldots$

SIGN

$\oplus$ IF $a_1 > 0$, $\ominus$ IF $a_1 < 0$.

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

ex. $4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \frac{64}{81} - \ldots$ ALTERNATING SERIES!

$$= \boxed{\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1}}_{\text{ALT. FACTOR}} \underbrace{4\left(\frac{2}{3}\right)^{n-1}}_{>0}} = \sum_{n=1}^{\infty} 4\left(\frac{-2}{3}\right)^{n-1} = \frac{4}{1-\left(-\frac{2}{3}\right)} = \frac{4}{5/3} = \frac{4\cdot 3}{5}$$

$$= 12/5$$

ALTERNATING SERIES TEST:

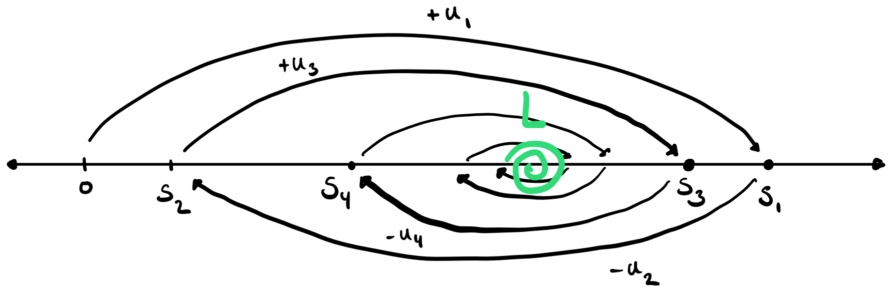
AN ALT. SERIES $\pm \sum_{n=1}^{\infty} (-1)^{n+1} u_n$, $u_n > 0$ CONVERGES IF

(1) u_n's ARE EVENTUALLY DECREASING.

i.e. $u_n \geq u_{n+1}$ FOR ALL $n \geq N$.

(2) $\lim_{n\to\infty} u_n = 0$.

IDEA:



$S_1 = u_1$

$S_2 = u_1 - u_2 \quad (u_1 > u_2)$

$S_3 = u_1 - u_2 + u_3 \quad (u_2 > u_3)$

ex. ALTERNATING HARMONIC SERIES: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

CONVERGES, BY ALT. SERIES TEST (1) $u_n = \frac{1}{n}$ DECREASES

(2) $\lim_{n \to \infty} \frac{1}{n} = 0$ ✓

ex. $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ $\cancel{= -\sum_{n=1}^{\infty} (-1)^{n+1} \frac{10^n}{(n+1)!}}$

ALTERNATING!

IF $\lim_{n \to \infty} \frac{10^n}{(n+1)!} = 0$ THEN SERIES CONVERGES (ALT. SERIES TEST)

IF $\lim_{n \to \infty} \frac{10^n}{(n+1)!} \neq 0$ THEN SERIES DIVERGES (DIVERGENCE TEST)

$u_1 = \frac{10^1}{(1+1)!} = \frac{10}{2} = 5$

$u_2 = \frac{10^2}{3!} = \frac{100}{6} \approx 16.7$

$u_3 = \frac{10^3}{4!} = \frac{1000}{24} \approx 41.7$

$u_n \left(\frac{10}{n+1} \right) = u_{n+1}$

$\frac{10^n}{n!} \left(\frac{10}{n+1} \right) = \frac{10^{n+1}}{(n+1)!}$

NOTE: $(n+1)! = (n+1)n!$

$$u_n\left(\frac{10}{n+1}\right)=u_{n+1}$$

$u_{10}\,\frac{10}{11} = u_{11}$ WHEN $n \geq 10$, $u_n > u_{n+1}$ DECREASING.

$u_{11}\,\frac{10}{12} = u_{12}$

$u_{12}\,\frac{10}{13} = u_{13}$

$$u_{11} = \frac{10}{11}u_{10}$$

$$u_{12} = \frac{10}{12}(u_{11}) = \left(\frac{10}{12}\right)\left(\frac{10}{11}\right)u_{10} \leq \left(\frac{10}{11}\right)^2 u_{10}$$

$$u_{13} = \frac{10}{13}(u_{12}) = \left(\frac{10}{13}\right)\left(\frac{10}{12}\right)\left(\frac{10}{11}\right)u_{10} \leq \left(\frac{10}{11}\right)^3 u_{10}$$

$$\vdots$$

$$u_{10+k} \leq u_{10}\left(\frac{10}{11}\right)^k$$

SINCE $0 < u_{10+k} \leq u_{10}\left(\frac{10}{11}\right)^k$

AND $\lim_{k\to\infty} u_{10}\left(\frac{10}{11}\right)^k = 0$, $\lim_{n\to\infty} u_n = \lim_{k\to\infty} u_{10+k} = 0$

(SQUEEZE THM).

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ CONVERGES BY ALT. SERIES TEST.

NOTE: $n!$ GROWS FASTER THAN a^n FOR ANY POSITIVE a!

ex. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(1+\frac{1}{n}\right)$ CONV. OR DIV.?

ALTERNATING! (1) $\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right] = \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right) < 0$ FOR $x \geq 1$

NEGATIVE DERIVATIVE

$\Rightarrow \ln\left(1+\frac{1}{n}\right)$ IS DECREASING SEQUENCE.

(2) $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(1) = 0$

$\Rightarrow$ CONVERGES BY ALT. SERIES TEST.

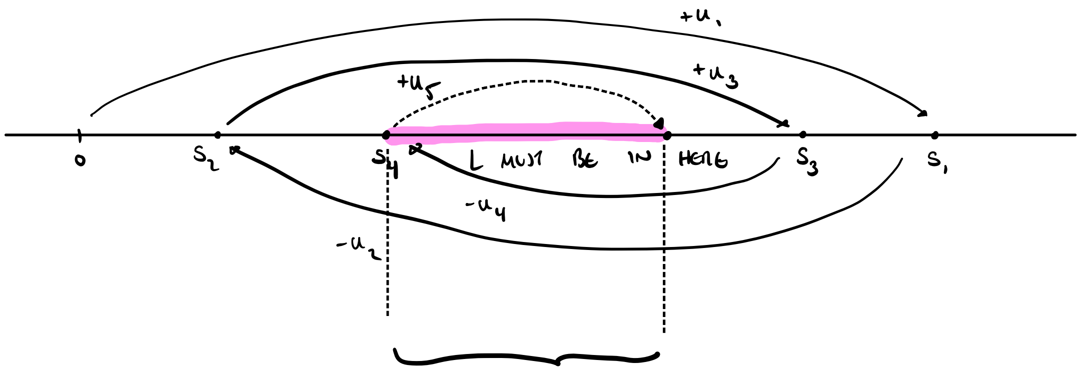
ALTERNATING SERIES ESTIMATION THM

SUPPOSE AN ALTERNATING SERIES CONVERGES.

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = L$$

WE CAN APPROXIMATE THE SUM OF THE SERIES

$$L \approx S_n = u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n+1} u_n$$ [n^{TH} PARTIAL SUM]



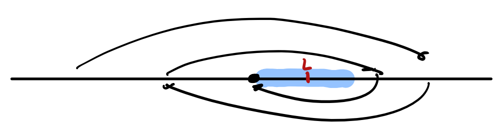
THE MAXIMUM ERROR IS u_{n+1} (NEXT TERM)

IF THE LAST TERM u_n IS POSITIVE THEN S_n IS AN OVERESTIMATE;
NEGATIVE UNDERESTIMATE.

ex. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

$$S_{30} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots - \frac{1}{30} \approx 0.67676$$

OVERESTIMATE OR UNDERESTIMATE



MAXIMUM ERROR : $\frac{1}{31}$