§11.4 The Comparison Theorems

Focus: Given a series $\sum_{n=1}^{\infty} a_n$, does it converge? (Yes/No)

If yes, we most likely won't know the value it converges to $\ddot{\frown}$

We can approximate the value with S_n (for large n).

The Direct Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ be two series such that $0 \le a_n \le b_n$ for all n.

Then

(1) If $\sum b_n$ converges then $\sum a_n$ converges

(2) If $\sum a_n$ diverges then $\sum b_n$ diverges.

(Similar to D.C.T. for improper int. $0 \le f(x) \le g(x)$

If $\int_1^{\infty} g(x)$ converges then $\int_1^{\infty} f(x)$ converges

If $\int_1^{\infty} f(x)$ diverges then $\int_1^{\infty} g(x)$ diverges)

Proof: Terms all non-negative, so sequence of partial sums is non-decreasing (monotonic)

Partial sums
$$\begin{cases} S_1 = a_1 \\ S_2 = a_1 + a_2 \\ S_3 = a_1 + a_2 + a_3 \\ \vdots \end{cases}$$

$\therefore$ Sequence of partial sums (series) converges if & only if they have an upper bound.

(1) FOR ANY n, WE HAVE

$$a_1 + a_2 + \dots + a_n \le b_1 + b_2 + \dots + b_n \le \sum_{n=1}^{\infty} b_n = M$$

$\therefore$ ALL PARTIAL SUMS $S_n = \sum_{k=1}^{n} a_k \le M$ UPPER BOUND.

$\therefore$ $\sum a_n$ CONVERGES ✓

(2) $$\underbrace{a_1 + a_2 + \dots + a_n}_{\text{UNBOUNDED AS } n \text{ INCREASES}} \le \underbrace{b_1 + b_2 + \dots + b_n}_{\text{MUST ALSO BE UNBOUNDED AS } n \text{ INCREASES.}}$$ ✓

$\therefore$ $\sum b_n$ DIVERGES. ■

DIVERGENCE TEST:

$$\lim_{n\to\infty} \frac{n-1}{n^4+2} = 0 \checkmark$$

WE CANNOT CONCLUDE THAT SERIES DIVERGES

ex. $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ CONVERGE OR DIVERGE?

FOR ALL $n = 1, 2, 3, \dots$

COMPARISON

$$0 \le \frac{n-1}{n^4+2} \le \frac{n}{n^4}$$

LARGER NUMERATOR

SMALLER DENOMINATOR

(BIGGER)

$n=10$ $\quad \frac{9}{10{,}002}$

$n=100$ $\quad \frac{99}{100{,}000{,}002}$

$$0 \le \frac{n-1}{n^4+2} \le \frac{1}{n^3}$$

P-SERIES TEST: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \le 1 \end{cases}$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots$$

CONVERGES
$p = 3 > 1$

$\therefore$ $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ ALSO CONVERGES BY DIRECT COMPARISON TEST. ✓

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ CONV OR DIV ?

HARMONIC SERIES $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \infty$ (INTEGRAL TEST)

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ DIVERGES

$0 \le \frac{1}{n3^n} \le \frac{1}{n}$

SMALLER DENOM
LARGER FRACTION

INCONCLUSIVE !

FIND SOMETHING BIGGER THAT CONVERGES
OR SOMETHING SMALLER THAT DIVERGES

$0 \le \frac{1}{n3^n} \le \frac{1}{3^n}$

SMALLER DENOM
LARGER FRACTION

$\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$

$=$ CONVERGES IF $|r| < 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1}$ CONVERGES

$r = \frac{1}{3} \Rightarrow |r| < 1$

$\therefore$ BY DIRECT COMP. TEST, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ ALSO CONVERGES.

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$

$\times\frac{1}{2}$ $\times\frac{1}{3}$ $\times\frac{1}{4}$ $\times\frac{1}{5}$

$\le 1 + \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}_{\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2} = 1 + 2 = 3$

$\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} \le 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ CONVERGES

$\therefore$ BY D.C.T. $\sum_{n=1}^{\infty} \frac{1}{n!}$ ALSO CONVERGES.

ex. $\sum_{n=1}^{\infty} \frac{2n+3}{4n^2-2n}$ CONV. OR DIV. ?

$\frac{2n+3}{4n^2-2n} \approx \frac{2n}{4n^2} = \frac{1}{2n}$

WHEN n IS LARGE

LARGER NUM.

$$0 \le \frac{1}{2n} = \frac{2n}{4n^2} \le \frac{2n+3}{4n^2-2n}$$

SMALLER DEN.

$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (p-TEST, HARMONIC SERIES DIV.)

$\therefore \sum_{n=1}^{\infty} \frac{2n+3}{4n^2-2n}$ ALSO DIVERGES BY D.C.T.

LIMIT COMPARISON TEST

SUPPOSE $a_n > 0$ & $b_n > 0$ FOR ALL $n = 1, 2, \cdots$.

(1) IF $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ $(0 < c < \infty)$

THEN $\sum a_n$ & $\sum b_n$ EITHER BOTH CONVERGE OR BOTH DIVERGE.

(2) IF $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ & $\sum b_n$ CONVERGES

THEN $\sum a_n$ ALSO CONVERGES

(3) IF $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ & $\sum b_n$ DIVERGES

THEN $\sum a_n$ ALSO DIVERGES.

PROOF: (i) SINCE $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, THERE EXISTS N S.T. $n \geq N$

WE HAVE $\left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}$

AND SO $-\frac{c}{2} \leq \frac{a_n}{b_n} - c \leq \frac{c}{2}$

$+c \qquad\qquad +c \qquad\qquad +c$

$$\frac{c}{2} \leq \frac{a_n}{b_n} \leq \frac{3c}{2}$$

$$\left(\frac{c}{2}\right) b_n \leq a_n \leq \left(\frac{3c}{2}\right) b_n$$

IF $\sum_{n=1}^{\infty} b_n$ DIVERGES, THEN

$\sum_{n=1}^{\infty} \left(\frac{c}{2}\right) b_n$ DIVERGES

$\Rightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES (D.C.T.)

IF $\sum_{n=1}^{\infty} b_n$ CONVERGES, THEN

$\sum_{n=1}^{\infty} \left(\frac{3c}{2}\right) b_n$ CONVERGES

$\Rightarrow \sum_{n=1}^{\infty} a_n$ CONVERGES BY D.C.T.

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ex. $\sum_{n=1}^{\infty} \frac{2n+3}{5n+4}$

$\lim_{n\to\infty} \frac{2n+3}{5n+4} = \frac{2}{5} \neq 0 \Rightarrow$ SERIES DIVERGES BY DIVERGENCE TEST.

ex. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

$\left(\frac{2n+3}{5n+4}\right)^n \approx \left(\frac{2}{5}\right)^n$

WHEN n IS LARGE.

LET $a_n = \left(\frac{2n+3}{5n+4}\right)^n$, $b_n = \left(\frac{2}{5}\right)^n$

NOTE : $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ CONVERGES. GEOMETRIC SERIES, $r = \frac{2}{5}$

$|r| < 1$ ✓

L.C.T. :

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{\left(\frac{2}{5}\right)^n} = \lim_{n\to\infty} \left(\frac{2n+3}{5n+4} \cdot \frac{5}{2}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n : 1^{\infty}$$ IND. FORM.

(L'HO)

$$= \lim_{n\to\infty} \operatorname{Exp}\left(\ln\left(\left(\frac{10n+15}{10n+8}\right)^n \right) \right)$$

$$= \operatorname{Exp}\left[\lim_{n\to\infty} \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{1/n} \right] : \frac{0}{0}$$

$$= \operatorname{Exp}\left[\lim_{n\to\infty} \frac{\frac{1}{10n+15}(10) - \frac{1}{10n+8}(10)}{-1/n^2} \right]$$

$$= \operatorname{Exp}\left[\lim_{n\to\infty} \frac{-10n^2}{10n+15} + \frac{10n^2}{10n+8} \right]$$

$$= \operatorname{Exp}\left[\lim_{n\to\infty} \frac{-10n^2(10n+8) + 10n^2(10n+15)}{(10n+15)(10n+8)} \right]$$

$$= \operatorname{Exp}\left[\lim_{n\to\infty} \frac{\cancel{-100n^3} - 80n^2 + \cancel{100n^3} + 15n^2}{100n^2 + 230n + 120} \right]$$

$$= \operatorname{Exp}\left[-\frac{65}{100} \right] = e^{-.65}$$ ← POSITIVE, FINITE # ✓

∴ SINCE $\sum \left(\frac{2}{5}\right)^n$ CONVERGES, SO DOES $\sum \left(\frac{2n+3}{5n+4}\right)^n$ BY L.C.T.