# § 11.3 Integral Test & Estimates of Sums

## Two Similar Questions:

Series $\sum_{n=1}^{\infty} a_n$ converge / diverge?

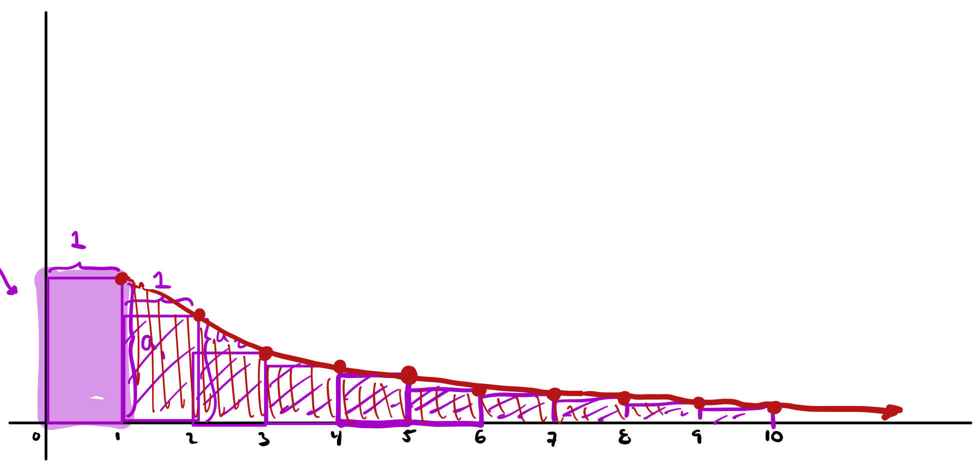
Improper Integral $\int_1^{\infty} f(x)\,dx$ converge / diverge?

Connection: Let $f(x)$ be a continuous function on $[1, \infty)$,

$f(x)$ positive, decreasing, $f(n) = a_n$ for $n = 1, 2, 3, \ldots$

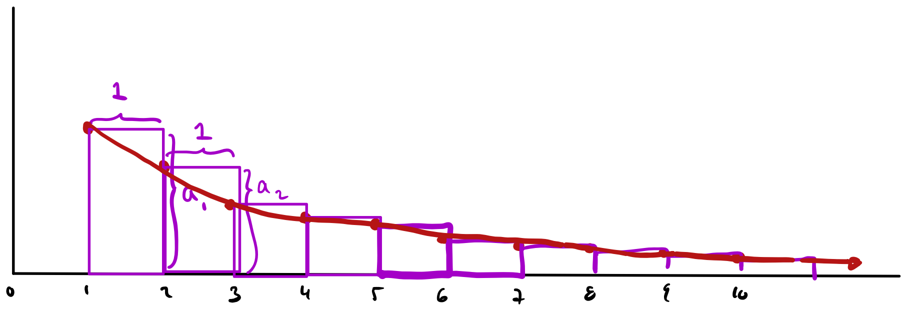
$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ , $a_n \geq 0$ for all $n$.

Series = total area of rect.




$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n \leq a_1 + \underbrace{\int_1^{\infty} f(x)\,dx}$$

So if this improper int. converges (finite) the the series must converge (finite).




$$\sum_{n=1}^{\infty} a_n \geq \underbrace{\int_1^{\infty} f(x)\,dx}$$

IF THIS IMPROPER INT. DIVERGES, THEN THE SERIES ALSO DIVERGES.

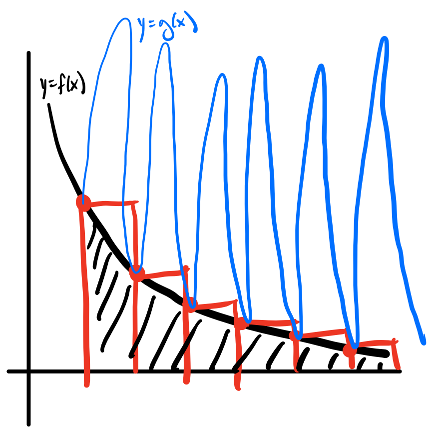
IN SUMMARY:

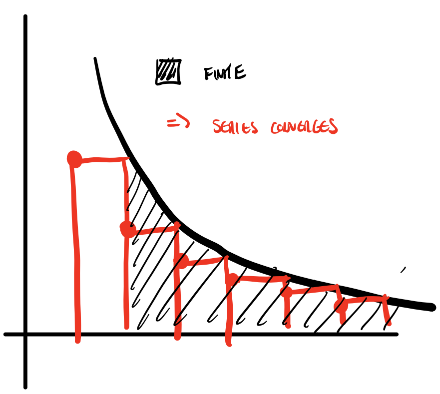
**The Integral Test** Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. *

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

* WARNING: $\displaystyle\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$ NUMERICALLY.



$f(n) = g(n) = a_n$ FOR $n = 1, 2, \dots$

AND IF $\int_1^\infty f(x)\,dx = \infty$ THEN $\sum_{n=1}^\infty a_n = \infty$ (BECAUSE $f$ IS DECREASING)

BUT SINCE $g(x)$ IS NOT DECREASING, YOU SEE IT IS POSSIBLE FOR $\int_1^\infty g(x) = \infty$

AND FOR $\sum a_n$ TO STILL CONVERGE.

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ex. **p - SERIES :** $\sum_{n=1}^\infty \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

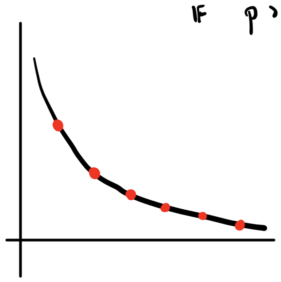
FOR WHAT VALUES OF $p$ DOES THIS SERIES CONVERGE?

DIVERGENCE TEST : IF $\lim_{n\to\infty} a_n \left( = \lim_{n\to\infty} \frac{1}{n^p} \right) \neq 0$ THEN SERIES DIVERGES.

IF $p \leq 0$ $(-p > 0)$ THEN $\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} = \infty$

IF $p \leq 0$ SERIES DIVERGES.

IF $p > 0$, CONSIDER $f(x) = \frac{1}{x^p}$, $x \geq 1$. $f(n) = a_n = \frac{1}{n^p}$ FOR $n = 1, 2, \dots$



$f(x) > 0$, $f'(x) = \frac{d}{dx}\left[x^{-p}\right] = -p\,x^{-p-1} < 0$ => $f$ DECREASING ✓

INTEGRAL TEST : $\sum_{n=1}^\infty \frac{1}{n^p}$ CONVERGES IF & ONLY IF $\int_1^\infty \frac{1}{x^p}\,dx$ CONVERGES

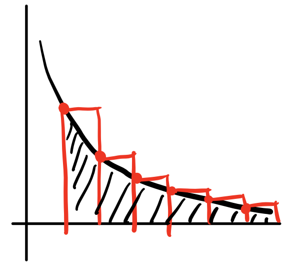
p-TEST : $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{CONVERGES WHEN } p > 1 \\ \text{DIVERGES WHEN } p \leq 1 \end{cases}$

$$\boxed{\text{p-TEST FOR SERIES } \sum_{n=1}^\infty \frac{1}{n^p} \begin{cases} \text{CONVERGES WHEN } p > 1 \\ \text{DIVERGES WHEN } p \leq 1 \end{cases}}$$

*

* WARNING : $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ FOR $p > 1$

BUT $\sum_{n=1}^{\infty} \frac{1}{n^p} \neq \frac{1}{p-1}$



FAMOUSLY : $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$

$$\int_1^\infty \frac{1}{x^2}\,dx = \frac{1}{2-1} = 1$$

ex. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$ CONVERGE OR DIVERGE ?

CONSIDER $f(x) = x^2 e^{-x/3}$ , $x \geq 1$

$f(x) > 0$ POSITIVE ✓

$f(x)$ IS DECREASING ? EVENTUALLY! ✓

$$f'(x) = 2x e^{-x/3} + x^2 e^{-x/3}\left(-\frac{1}{3}\right) = x e^{-x/3}\left(2 - \frac{1}{3}x\right)$$

| | | 0 | | 6 | |
|---|---|---|---|---|---|
| $f'$ | $-$ | | $+$ | | $-$ |
| $x$ | $-$ | | $+$ | | $+$ |
| $e^{-x/3}$ | $+$ | | $+$ | | $+$ |
| $\left(2 - \frac{1}{3}x\right)$ | $+$ | | $+$ | | $-$ |

$f(x)$ IS DECREASING WHEN $x \geq 6$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$ CONVERGES $\Longleftrightarrow$ $\int_1^\infty x^2 e^{-x/3}\,dx$ CONVERGES .

$$\int_1^{\infty} x^2 e^{-x/3}\,dx = \lim_{t\to\infty} \Big[ -3x^2 e^{-x/3} - 18x e^{-x/3} - 54 e^{-x/3} \Big]_1^t$$

$u = x^2$ $\quad v = -3e^{-x/3}$

$du = 2x\,dx$ $\quad dv = e^{-x/3}\,dx$

$u = x$ $\quad v = -3e^{-x/3}$

$du = dx$ $\quad dv = e^{-x/3}\,dx$

$$\rightarrow uv - \int v\,du = -3x^2 e^{-x/3} + 6 \int x e^{-x/3}\,dx$$

$$= -3x^2 e^{-x/3} + 6\left[-3xe^{-x/3} + 3\int e^{-x/3}\,dx\right]$$

$$= -3x^2 e^{-x/3} + 6\left[-3xe^{-x/3} - 9e^{-x/3}\right]$$

$$= -3x^2 e^{-x/3} - 18xe^{-x/3} - 54e^{-x/3} \quad \checkmark$$

$$= \underbrace{\lim_{t\to\infty} \frac{-3t^2 - 18t - 54}{e^{t/3}}} + \underbrace{e^{-1/3}(3 + 18 + 54)}_{\text{FINITE}}$$

$$\lim_{t\to\infty} \frac{-3t^2 - 18t - 54}{e^{t/3}} = \frac{\infty}{\infty} \qquad \text{L'HÔ}$$

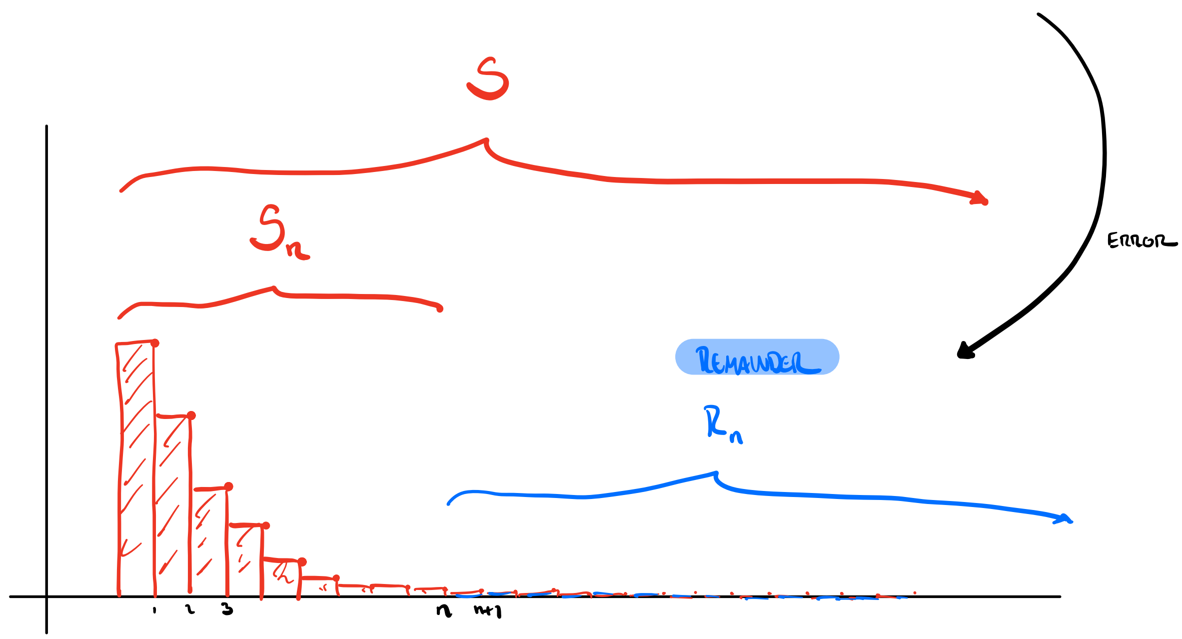
$$\lim_{t\to\infty} \frac{-6t - 18}{\frac{1}{3}e^{t/3}} = \frac{\infty}{\infty} = \lim_{t\to\infty} \frac{-6}{\frac{1}{9}e^{t/3}} = 0$$

LIMIT EXISTS $\rightarrow$ $\underbrace{\text{IMP. INT. CONV.} \rightarrow \text{SERIES CONVERGES}}_{\text{INT. TEST.}}$

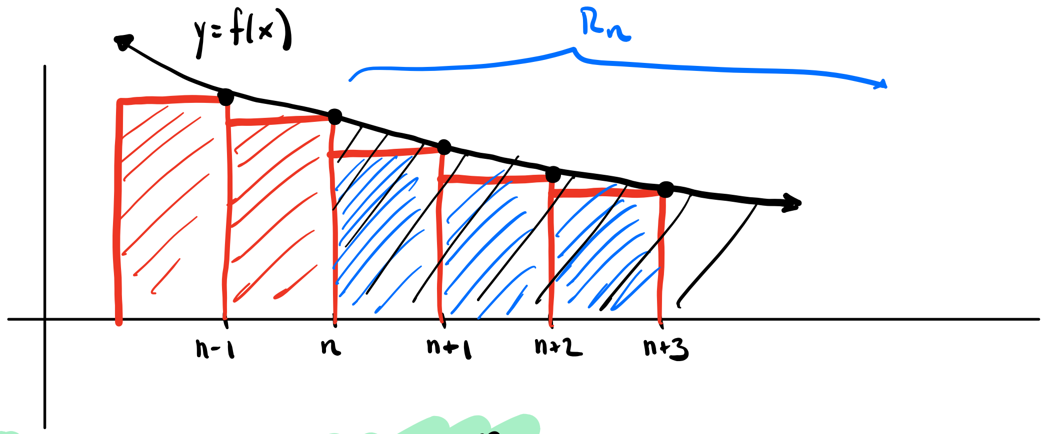
# ESTIMATING A SERIES $\sum_{n=1}^{\infty} a_n \approx \sum_{k=1}^{n} a_k$

e.g. $\quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \approx 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{100^2}$

BUT HOW CLOSE IS THE ESTIMATE?

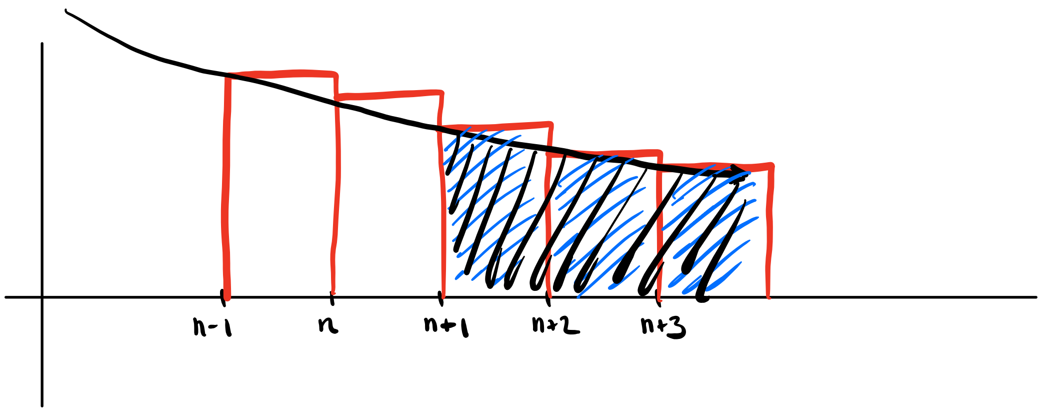



$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$




$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

where $f(n) = a_n$




**2 Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where $f$ is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

## EXAMPLE 5

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
(b) How many terms are required to ensure that the sum is accurate to within 0.0005?