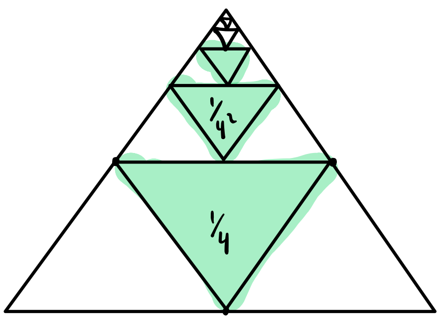
# § 11.2 Series

INFINITE SEQUENCE : $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \ldots = \left\{ \frac{1}{4^n} \right\}_{n=1}^{\infty}$

INFINITE SERIES : $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{4^n}$

EQUILATERAL TRIANGLE



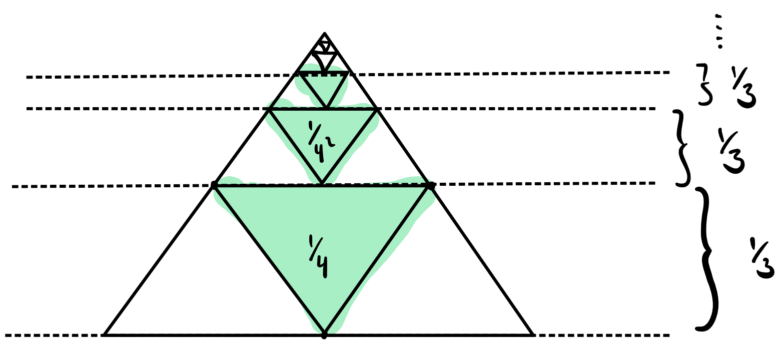

AREA 1

AREA ▮ $= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$

INCREASING , BOUNDED

$\Rightarrow$ CONVERGES !

SERIES (SUM) CONVERGES




1/3 OF THE TRIANGLE IS SHADED

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$$

| n | a_n | s_n |
|---|---|---|
| 1 | 0.25 | 0.25 |
| 2 | 0.0625 | 0.3125 |
| 3 | 0.015625 | 0.328125 |
| 4 | 0.00390625 | 0.33203125 |
| 5 | 0.0009765625 | 0.3330078125 |
| 6 | 0.000244140625 | 0.3332519531 |
| 7 | 0.00006103515625 | 0.3333129883 |
| 8 | 0.00001525878906 | 0.3333282471 |
| 9 | 0.000003814697266 | 0.3333320618 |
| 10 | 0.0000009536743164 | 0.3333330154 |
| 11 | 0.0000002384185791 | 0.3333332539 |
| 12 | 0.00000005960464478 | 0.3333333135 |
| 13 | 0.00000001490116119 | 0.3333333284 |
| 14 | 0.000000003725290298 | 0.3333333321 |
| 15 | 0.0000000009313225746 | 0.333333333 |
| 16 | 0.0000000002328306437 | 0.3333333333 |

**2 Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let $s_n$ denote its $n$th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number $s$ is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

"SERIES ARE SEQUENCES OF PARTIAL SUMS"

## GEOMETRIC SERIES

DEF: FOR ANY $a \neq 0$,

$$\sum_{n=1}^{\infty} ar^{n-1} = a \overset{\times r}{+} ar \overset{\times r}{+} ar^2 \overset{\times r}{+} ar^3 + \cdots$$

IS A GEOMETRIC SERIES.

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WHAT CAN WE SAY ABOUT CONVERGENCE / DIVERGENCE OF GEO. SERIES?

(1) IF $r = 1$: THEN $\sum_{n=1}^{\infty} ar^{n-1} = a + a + a + a + \ldots$ DIVERGES.

(2) IF $r \neq 1$: LET $S_n = \sum_{k=1}^{n} ar^{k-1}$ $n^{TH}$ PARTIAL SUM.

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + \cancel{ar^{n-1}}$$

$$-\left( rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\cdots} + ar^n \right)$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}} = \sum_{k=1}^{n} ar^{k-1}$$

BY DEF,

$$\sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} \sum_{k=1}^{n} ar^{k-1} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \quad \begin{cases} \text{CONVERGES TO } \dfrac{a}{1-r} & \text{IF } |r| < 1 \\ \text{DIVERGES IF } |r| > 1 \text{ OR } r = -1 \end{cases}$$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{IF } |r| < 1 \\ \text{DIVERGES IF } r = -1 \\ \text{DIVERGES IF } |r| > 1 \end{cases}$$

IN SUMMARY:

**4** The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots \quad , a \neq 0$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

ex. For which values of $x$ does the series

$$\sum_{n=1}^{\infty} \frac{2^{2n} x^n}{5^{n-1}}$$

converge? What does it converge to?

Re index!

$$= 4x + \frac{16x^2}{5} + \frac{64x^3}{5^2} + \frac{256x^4}{5^3}$$

$\cdot \frac{4x}{5}$ $\cdot \frac{4x}{5}$ $\cdot \frac{4x}{5}$

Rewrite as a geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r| < 1$$

$$\sum_{n=1}^{\infty} 4x\left(\frac{4x}{5}\right)^{n-1} \text{ converges to } \frac{4x}{1-\frac{4x}{5}} \quad \text{if } \left|\frac{4x}{5}\right| < 1$$

i.e. $-1 < \frac{4x}{5} < 1 \implies -5 < 4x < 5$

$\implies -\frac{5}{4} < x < \frac{5}{4}$

$$\boxed{\text{converges to } \frac{4x}{1-\frac{4x}{5}} \text{ if } -\frac{5}{4} < x < \frac{5}{4}}$$

## Repeating Decimals $\iff$ Rational Number

ex. Write the number $5.8\overline{305}$ as a fraction.

5.8 305 305 305 305

$$5.8 + \underbrace{\frac{305}{10000} + \frac{305}{10{,}000{,}000} + \frac{305}{10{,}000{,}000{,}000}}_{\text{Geometric Series!}} + \dots$$

$\cdot \frac{1}{1000}$

$$5.8 + \sum_{n=1}^{\infty} \underbrace{\frac{305}{10{,}000}}_{a}\underbrace{\left(\frac{1}{1000}\right)^{n-1}}_{r^{n-1}} = 5.8 + \frac{\frac{305}{10{,}000}}{1-\frac{1}{1000}} = 5.8 + \frac{305}{10{,}000}\cdot\frac{1000}{999}$$

$$\frac{a}{1-r} \qquad = \frac{58}{10} + \frac{305}{9990} = \frac{582470}{99900}$$

## Telescoping Sums

ex. Use partial fractions to show that $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ converges.

$$\left( \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \frac{2}{k(k+2)}}_{S_n} = \lim_{n\to\infty} S_n \quad \text{exist} \right)$$

$n^{th}$ partial sum $\quad S_n$

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \quad \Rightarrow \quad 2 = A(n+2) + Bn$$

$$n=0: \quad 2 = 2A \Rightarrow A = 1$$

$$n=-2: \quad 2 = -2B \Rightarrow B = -1$$

Telescoping sum

$$\therefore \sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+2} = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}\right)$$

$$+ \dots + \left(\cancel{\tfrac{1}{n-2}} - \cancel{\tfrac{1}{n}}\right) + \left(\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n+1}\right) + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+2}\right)$$

$$= \lim_{n\to\infty} 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = 1 + \frac{1}{2} - 0 - 0 = \boxed{\frac{3}{2}}$$

**6 Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

<u>PROOF:</u> RECALL $n^{TH}$ PARTIAL SUM $S_n = \sum_{k=1}^{n} a_k$, $S = \lim_{n\to\infty} S_n$ (SUM OF SERIES)

THEN $a_n = (a_1 + \dots + a_{n-1} + a_n) - (a_1 + \dots + a_{n-1}) = S_n - S_{n-1}$

SINCE $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = \sum_{n=1}^{\infty} a_n$ ( $= S$ SUM OF SERIES )

$\Rightarrow \lim_{n\to\infty} S_n - S_{n-1} = S - S = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ ■

DEAL BREAKER! FOR CONVERGENCE!

**7 Test for Divergence** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$\lim_{n\to\infty} a_n = 0$ NECESSARY BUT NOT SUFFICIENT FOR SERIES TO CONVERGE.

WARNING! JUST BECAUSE $\lim_{n\to\infty} a_n = 0$, IT DOES <u>NOT</u> MEAN THE SERIES CONVERGES.

IT MEANS IT <u>MIGHT</u> CONVERGE.

BUT IF $\lim_{n\to\infty} a_n \neq 0$, THEN SERIES DOES <u>NOT</u> CONVERGE ( IT DIVERGES ).

<u>ex.</u> **36.** $\sum_{n=1}^{\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n}$ CONVERGE OR DIVERGE?

DO THE TERMS GO TO 0? $\lim_{n\to\infty} a_n = 0$ ?

$$\lim_{n\to\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n} = \frac{1}{1 + 0} = 1 \neq 0$$

$\Rightarrow$ SERIES DIVERGES BY "DIVERGENCE TEST".

## Harmonic Series

Converge or Diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{15} + \frac{1}{16}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16} + \frac{1}{16} + \dots$$

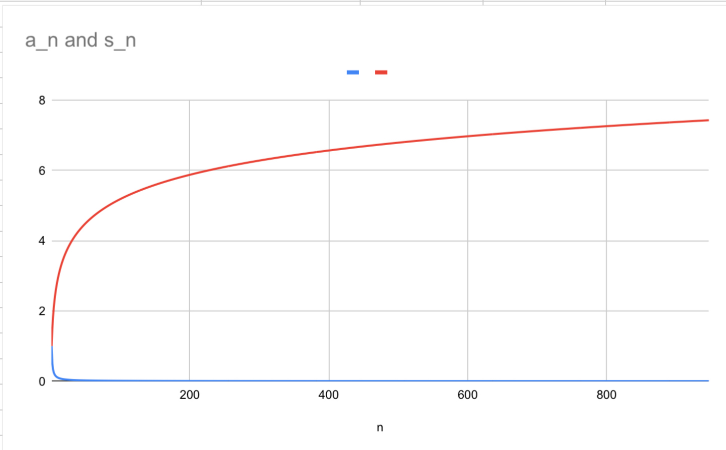
$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

$\therefore$ Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges

Harmonic Series

| n | a_n | s_n |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 0.5 | 1.5 |
| 3 | 0.3333333333 | 1.833333333 |
| 4 | 0.25 | 2.083333333 |
| 5 | 0.2 | 2.283333333 |
| 6 | 0.1666666667 | 2.45 |
| 7 | 0.1428571429 | 2.592857143 |
| 8 | 0.125 | 2.717857143 |
| 9 | 0.1111111111 | 2.828968254 |
| 10 | 0.1 | 2.928968254 |
| 11 | 0.09090909091 | 3.019877345 |
| 12 | 0.08333333333 | 3.103210678 |
| 13 | 0.07692307692 | 3.180133755 |
| 14 | 0.07142857143 | 3.251562327 |
| 15 | 0.06666666667 | 3.318228993 |
| 16 | 0.0625 | 3.380728993 |
| 17 | 0.05882352941 | 3.439552523 |
| 18 | 0.05555555556 | 3.495108078 |
| 19 | 0.05263157895 | 3.547739657 |
| 20 | 0.05 | 3.597739657 |
| 21 | 0.04761904762 | 3.645358705 |
| 22 | 0.04545454545 | 3.69081325 |
| 23 | 0.04347826087 | 3.734291511 |
| 24 | 0.04166666667 | 3.775958178 |
| 25 | 0.04 | 3.815958178 |
| 26 | 0.03846153846 | 3.854419716 |
| 27 | 0.03703703704 | 3.891456753 |




**8 Theorem** If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where $c$ is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Limit Rules

(Series are limits of partial sums)

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$$

$$S_n = \sum_{k=1}^{n} a_k$$

ex. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{2}{n(n+2)}\right)$ CONVERGENT OR DIVERGENT? IF CONVERGENT, WHAT IS THE SUM OF THE SERIES?

$$= \sum_{n=1}^{\infty}\frac{1}{e^n} + \sum_{n=1}^{\infty}\frac{2}{n(n+2)}$$

$$\frac{3}{2}$$

$$\frac{1}{e}+\left(\frac{1}{e}\right)^2+\left(\frac{1}{e}\right)^3+\left(\frac{1}{e}\right)^4+\ldots$$

$\times \frac{1}{e}$

GEOMETRIC SERIES: $a + ar + ar^2 + ar^3 + \ldots$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$ IF $|r|<1$, OTHERWISE SERIES DIVERGES.

$$\sum_{n=1}^{\infty}\frac{1}{e^n} = \frac{1}{e}+\left(\frac{1}{e}\right)^2+\left(\frac{1}{e}\right)^3+\left(\frac{1}{e}\right)^4+\ldots = \sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1}$$

$= \dfrac{1/e}{1-1/e}$ CONVERGES BECAUSE $|r| = \left|\frac{1}{e}\right| < 1$ ✓

$$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{2}{n(n+2)}\right) = \frac{1/e}{1-1/e} + \frac{3}{2}$$

$$= \boxed{\frac{1}{e-1}+\frac{3}{2}}$$