§11.1 SEQUENCES

REAL NUMBERS

Def: A SEQUENCE IS AN ORDERED LIST OF OBJECTS (CALLED TERMS).

$$a_1 \,,\, a_2 \,,\, a_3 \,,\, \dots \,,\, a_{n-1} \,,\, a_n \,,\, a_{n+1} \,,\, \dots$$

1^{st} TERM

2^{nd} TERM

PREVIOUS TO n^{TH} TERM

n^{TH} TERM

AFTER n^{TH} TERM

A SEQUENCE IS A FUNCTION. DOMAIN OF THE FUNCTION IS $\mathbb{N}$

NATURAL NUMBERS $1, 2, 3, \dots$

DOMAIN x	1	2	3	...
SEQUENCE $f(x)$	a_1	a_2	a_3	...

"DISCRETE" NOT "CONTINUOUS"

ex. $\left\{ \dfrac{n-1}{n+1} \right\}_{n=1}^{\infty}$: $a_n = \dfrac{n-1}{n+1}$ FOR $n = 1, 2, 3, \dots$ (MIRRORS FUNCTION NOTATION)

CURLY BRACKETS CONTAIN FORMULA FOR a_n

SIMILAR TO SUMMATION NOTATION $\sum_{n=1}^{\infty}$

THIS GIVES START & END VALUE.

THIS IS AN INFINITE SEQUENCE.

$$\left\{ \frac{n-1}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \dots \right\}$$

ex. $$\left\{ (-1)^n \sqrt{n} \right\}_{n=1}^{\infty} = \left\{ (-1)^1\sqrt{1}, (-1)^2\sqrt{2}, (-1)^3\sqrt{3}, \dots \right\}$$

$$= \left\{ \underset{\text{NEG}}{-\sqrt{1}}, \underset{\text{POS}}{\sqrt{2}}, \underset{\text{NEG}}{-\sqrt{3}}, \underset{\text{POS}}{\sqrt{4}}, \dots \right\}$$

"ALTERNATING SEQUENCE"

SIGN OF TERMS : POS $\rightleftarrows$ NEG

ex. FIND A FORMULA FOR THE GENERAL TERM a_n OF THE SEQUENCE

n	1	2	3	4	5
a_n	$\frac{1}{3}$,	$-\frac{3}{6}$,	$\frac{5}{9}$,	$-\frac{7}{12}$,	$\frac{9}{15}$, ...

SETUP

ALTERNATES: 1, -1, 1, -1, 1, ...

$(-1)^{n+1}$

NUMERATORS: 1 3 5 7 9

$(2n-1)$

DENOMINATORS: 3 6 9 12

$(3n)$

$$a_n = \frac{(-1)^{n+1}(2n-1)}{3n}$$

ex. SUPPOSE $a_1 = 1$, $a_2 = 1$, AND FOR $n \geq 3$ $a_n = a_{n-2} + a_{n-1}$.

LIST THE FIRST 8 TERMS OF THIS RECURSIVELY DEFINED SEQUENCE

n	1	2	3	4	5	6	7	8	...
a_n	1	1	2	3	5	8	13	21	...

$a_n = a_{n-2} + a_{n-1}$. FIBONACCI SEQUENCE.

$n=3$: $a_3 = a_1 + a_2$

$n=4$: $a_4 = a_2 + a_3$

Q: WHAT IS a_{50}?

WE NEED TO FIRST FIND $a_1 - a_{49}$

$a_{50} = a_{48} + a_{49}$

$a_{48} = a_{46} + a_{47}$

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

IF YOU FAR ENOUGH INTO THE SEQUENCE, ALL TERMS ARE CLOSE TO L.

SEQUENCE HAS A LIMIT $\Longleftrightarrow$ SEQUENCE CONVERGES.

$\underbrace{\text{SEQUENCE DOESN'T HAVE A LIMIT}}_{\text{SPECIAL CASE } \lim_{n \to \infty} a_n = \infty}$ $\Longleftrightarrow$ SEQUENCE DIVERGES.

TECHNICAL DEFINITION

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

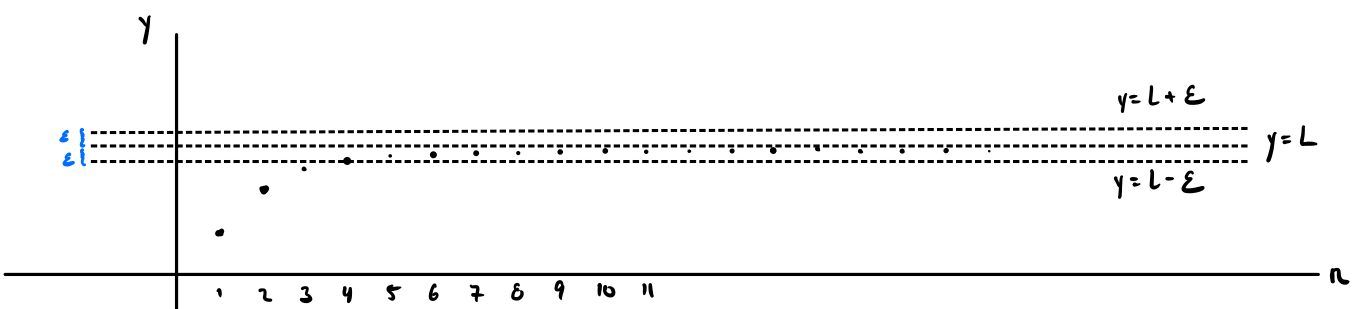
$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

HOW CLOSE TO L

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

HOW FAR OUT IN THE SEQUENCE



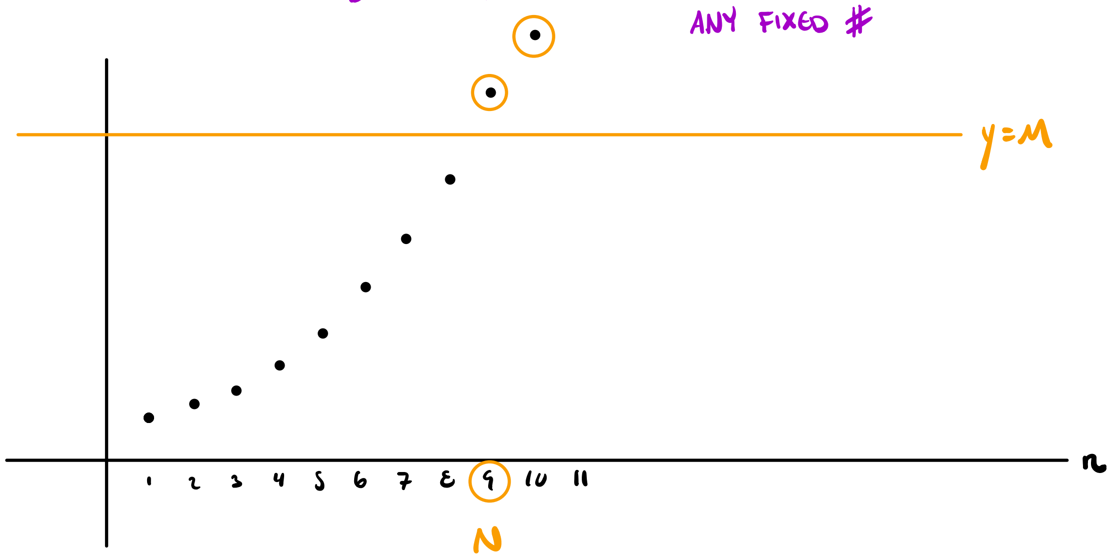
IF $n > 5$ THEN $|a_n - L| < \varepsilon$

5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M$$

EVENTUALLY

THE TERMS EXCEED ANY FIXED #



CONNECTION BETWEEN FUNCTIONS OF A REAL VARIABLE (e.g. DOMAIN $[1, \infty)$)

& SEQUENCES:

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $\boxed{f(n) = a_n}$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

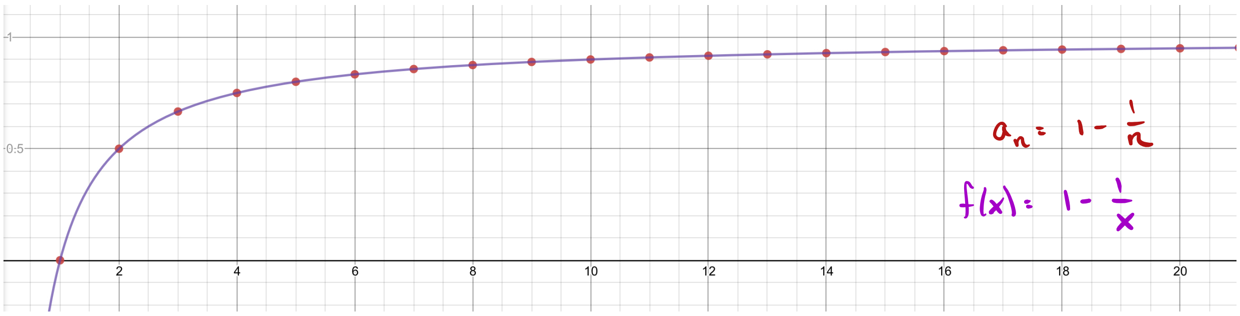
$f(x)$ IS DEFINED FOR ALL REAL #'S, AND $f(n) = a_n$

INTEGER

ex. $\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

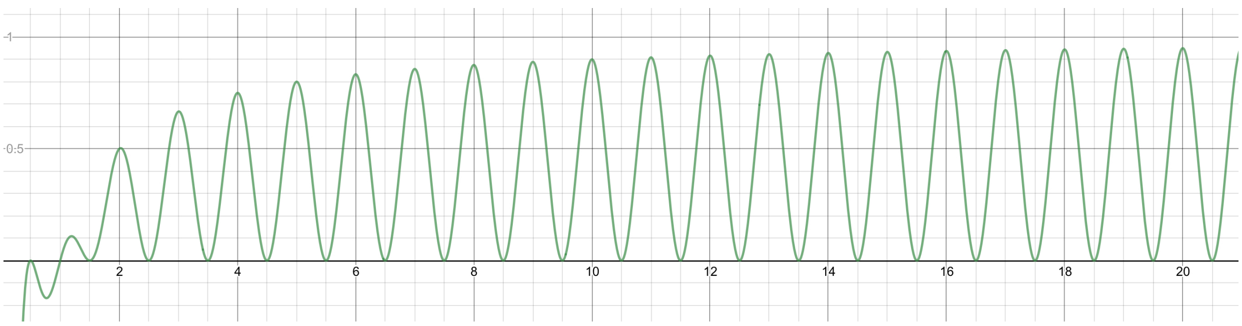
LET $f(x) = 1 - \frac{1}{x}$. NOTE: $f(n) = 1 - \frac{1}{n} = a_n$

SINCE $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n\to\infty} a_n = 1$

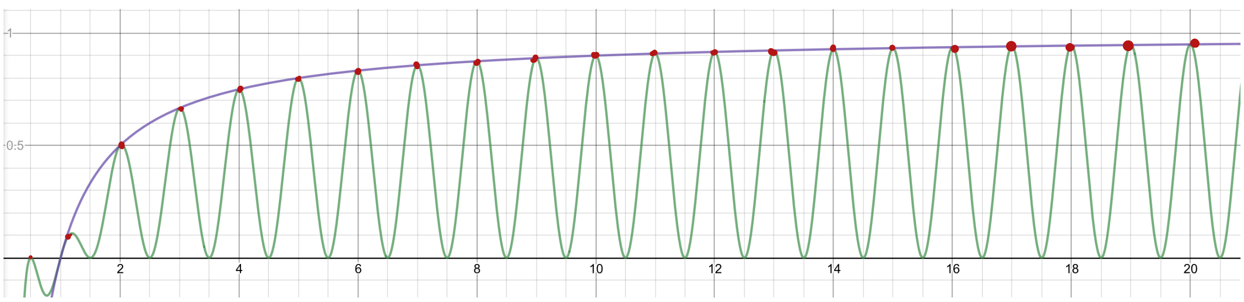


FUNCTION CONVERGES $\Rightarrow$ SEQUENCE CONVERGES

f $\qquad a_n = f(n)$



$y = \left(1 - \frac{1}{x}\right)\cos^2(\pi x)$



$\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)\cos^2(\pi x)$ DIVERGES.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

SEQUENCE $\quad r \xrightarrow{\times r} r^2 \xrightarrow{\times r} r^3 \xrightarrow{\times r} r^4, \ldots$ GEOMETRIC SEQUENCE.

IF $r = 1$: $\underset{1}{r} = \underset{1^2}{r^2} = \underset{1^3}{r^3} = \ldots = 1 \Rightarrow \lim\limits_{n\to\infty} r^n = 1$

IF $0 < r < 1$ THEN $r = \frac{1}{s}$, $s > 1$

$r \quad r^2 \quad r^3 \quad r^4 \longrightarrow 0$ ✓

$\frac{1}{s} \quad \frac{1}{s^2} \quad \frac{1}{s^3} \quad \frac{1}{s^4} \longrightarrow 0$

IF $r = 0 \Rightarrow r, r^2, \ldots r^n \to 0$

IF $-1 < r < 0$ . . .

ex. WHAT ABOUT $\{r^{1/n}\}$? $\{r^{-n}\}$?

$\lim\limits_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim\limits_{n\to\infty} r^{1/n} = \lim\limits_{x\to 0} r^x = r^0 = 1$

$$\lim_{n\to\infty} r^{-n} = \lim_{n\to\infty} \frac{1}{r^n} = \begin{cases} 0 & \text{IF } |r| > 1 \\ 1 & \text{IF } r = 1 \\ \text{DIVERGES OTHERWISE.} & \end{cases}$$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

LIMIT RULES FOR SEQUENCES.

SAME AS LIMIT RULES FOR FUNCTIONS.

SANDWICH THEOREM.

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

PROOF: $\lim_{n\to\infty} a_n = L \Rightarrow \forall \varepsilon > 0, \ \exists N_1$ s.t. $n \geq N_1 \Rightarrow |a_n - L| < \varepsilon$.

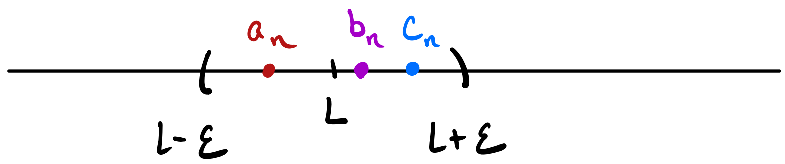
FOR ANY $\varepsilon > 0$, THERE EXISTS N_1 SUCH THAT

IF $n \geq N_1$ THEN $|a_n - L| < \varepsilon$

$\lim_{n\to\infty} c_n = L \Rightarrow \forall \varepsilon > 0, \ \exists N_2$ s.t. $n \geq N_2 \Rightarrow |c_n - L| < \varepsilon$.

THEN $\forall \varepsilon > 0$, IF $n \geq \max(N_1, N_2)$ THEN

$|a_n - L| < \varepsilon \qquad |c_n - L| < \varepsilon$



SINCE $a_n \leq b_n \leq c_n \Rightarrow |b_n - L| < \varepsilon$ ∎

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

(COROLLARY)

$$-|a_n| \le a_n \le |a_n|$$

$$\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0 \qquad \lim_{n\to\infty} |a_n| = 0$$

(GIVEN

$$\lim_{n\to\infty} a_n = 0 \quad \blacksquare$$

SQUEEZE THM
SANDWICH THM.

7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

$$\lim_{x\to L} f(x) = f(L)$$

DEF. OF CONTINUITY.

ex. $\lim_{n\to\infty} \sqrt{\dfrac{3n^2+1}{2n^2-1}}$

$$= \sqrt{\lim_{n\to\infty} \frac{3n^2+1}{2n^2-1}} = \sqrt{\frac{3}{2}}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

a_n MONOTONIC IF EITHER $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \ldots$ NON - DECREASING

MORE GENERAL $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots$ NON - INCREASING

ex. SHOW THAT THE SEQUENCE $\left\{ \sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \right\}$ IS INCREASING.

$2^{1/2}$ $\quad (2 \cdot 2^{1/2})^{1/2}$ $\quad \left(2(2 \cdot 2^{1/2})^{1/2}\right)^{1/2}$

$2^{1/2}$ $\quad 2^{3/4}$ $\quad 2^{7/8}$

EXPONENTS INCREASING.

2^x IS INCREASING

$\frac{d}{dx}\left[2^x\right] = 2^x \ln(2) > 0$

$f(x) = 2^x$, $f(n) = n$

SINCE f IS INCREASING, a_n IS INCREASING. ✓

§11.1 SEQUENCES (CONTINUED)

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

M UPPER BOUND (CEILING)

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

m LOWER BOUND (FLOOR)

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

ex. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

UPPER BOUND : 23, 55, ∞ MANY UPPER BOUNDS

1 LEAST UPPER BOUND

LOWER BOUND : -100, -200

0 GREATEST LOWER BOUND

ex. SHOW THAT THE SEQUENCE $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ IS BOUNDED.

NOTE: THIS IS A RECURSIVE SEQ.

$a_1 = \sqrt{2}$, $a_n = \sqrt{2 + a_{n-1}}$

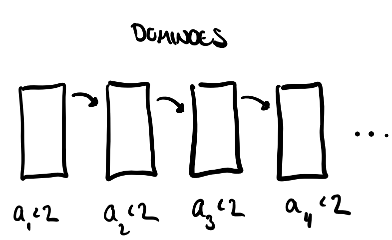
$a_1 < 2$

IF $a_{n-1} < 2$ THEN $2 + a_{n-1} < 4$

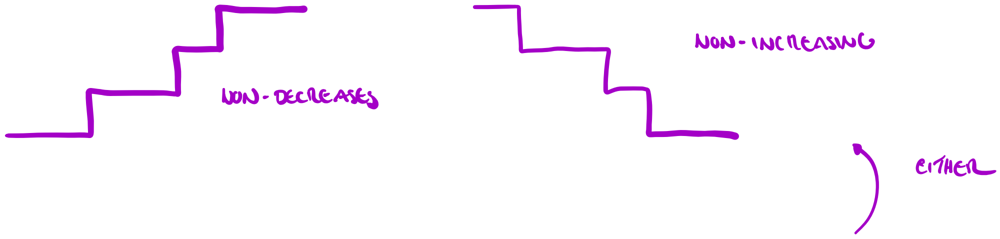
AND SO $a_n = \sqrt{2 + a_{n-1}} < \sqrt{4} = 2$

IF $a_{n-1} < 2$ THEN $a_n < 2$ (REPEAT)

" INDUCTION "

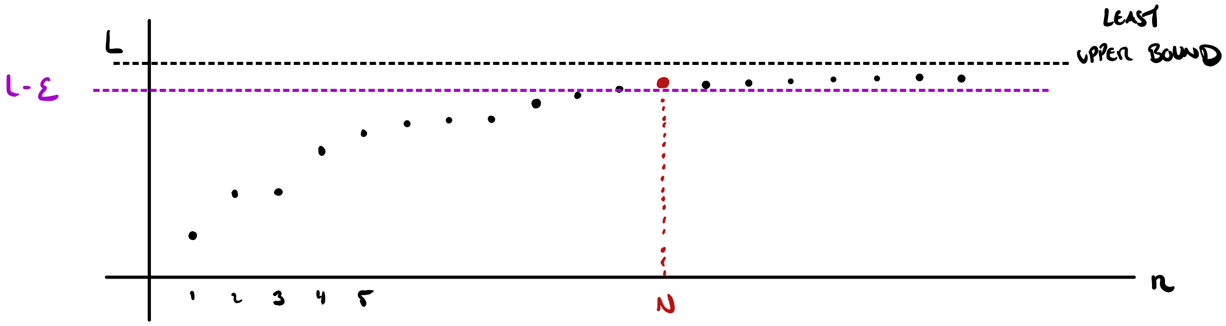


$\therefore$ ALL TERMS $a_n < 2$. 2 IS AN UPPER BOUND.



12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

<u>PROOF</u>: SUPPOSE $\{a_n\}$ IS NON-DECREASING $a_{n-1} \leq a_n$ & BOUNDED ABOVE.



THEN THERE EXISTS A LEAST UPPER BOUND, L.

FOR ALL $\varepsilon > 0$ THERE EXISTS N S.T. $a_N > L - \varepsilon$

NON-DECREASING $\Rightarrow$ $a_n > L - \varepsilon$ FOR $n \geq N$

$\Rightarrow$ $L - a_n < \varepsilon$

$\Rightarrow$ $|L - a_n| < \varepsilon$ $\Rightarrow$ $\lim_{n \to \infty} a_n = L$. ∎

ex. FIND THE $\lim_{n\to\infty} a_n$ WHERE $a_1 = \sqrt{2}$

$$a_2 = \sqrt{2+\sqrt{2}}$$

$$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\vdots$$

	A	B
1	n John Adamski	a_n
2	1	1.414213562
3	2	1.847759065
4	3	1.961570561
5	4	1.990369453
6	5	1.997590912
7	6	1.999397637
8	7	1.999849404
9	8	1.999962351
10	9	1.999990588

LET $x = \lim_{n\to\infty} a_n = \sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$

THEN $\sqrt{2+x} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} = x$

$$\sqrt{2+x} = x \implies 2+x = x^2 \implies x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1-4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{9}}{2} = 2,\ -1$$

$\otimes$