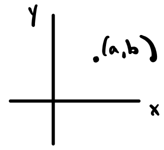
§ 10.3 Polar Coordinates

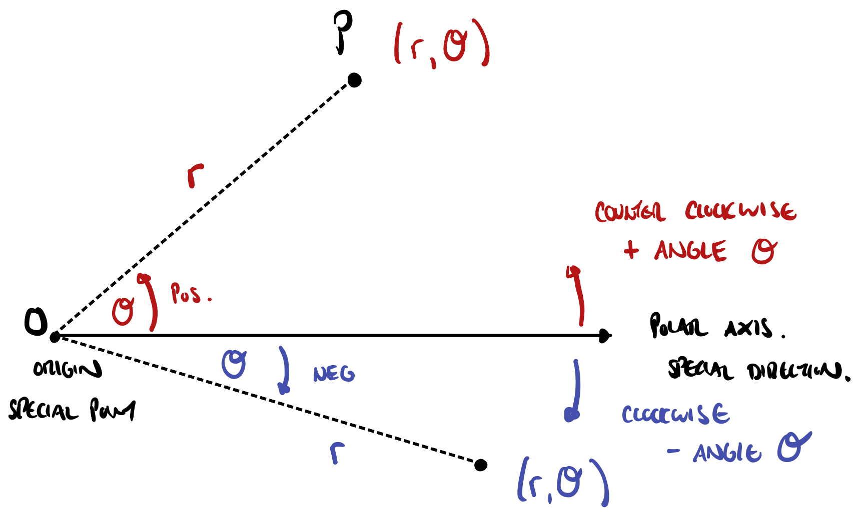
Cartesian Coord. System



(a,b) : $b = f(a)$

New Coord System

Origin, Polar Axis
↓
Position is relative.



2 #'s to describe location of points P

$|r|$: distance to origin *

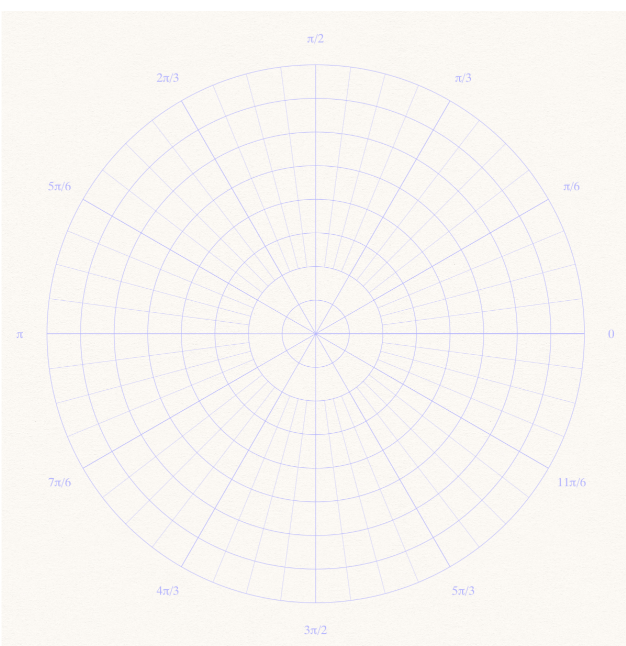
θ: angle that $\overline{OP}$ makes with polar axis. (oriented +/-)

* r is oriented.

Given an angle θ,

If $r > 0$ → walk fwd distance $|r|$ to P

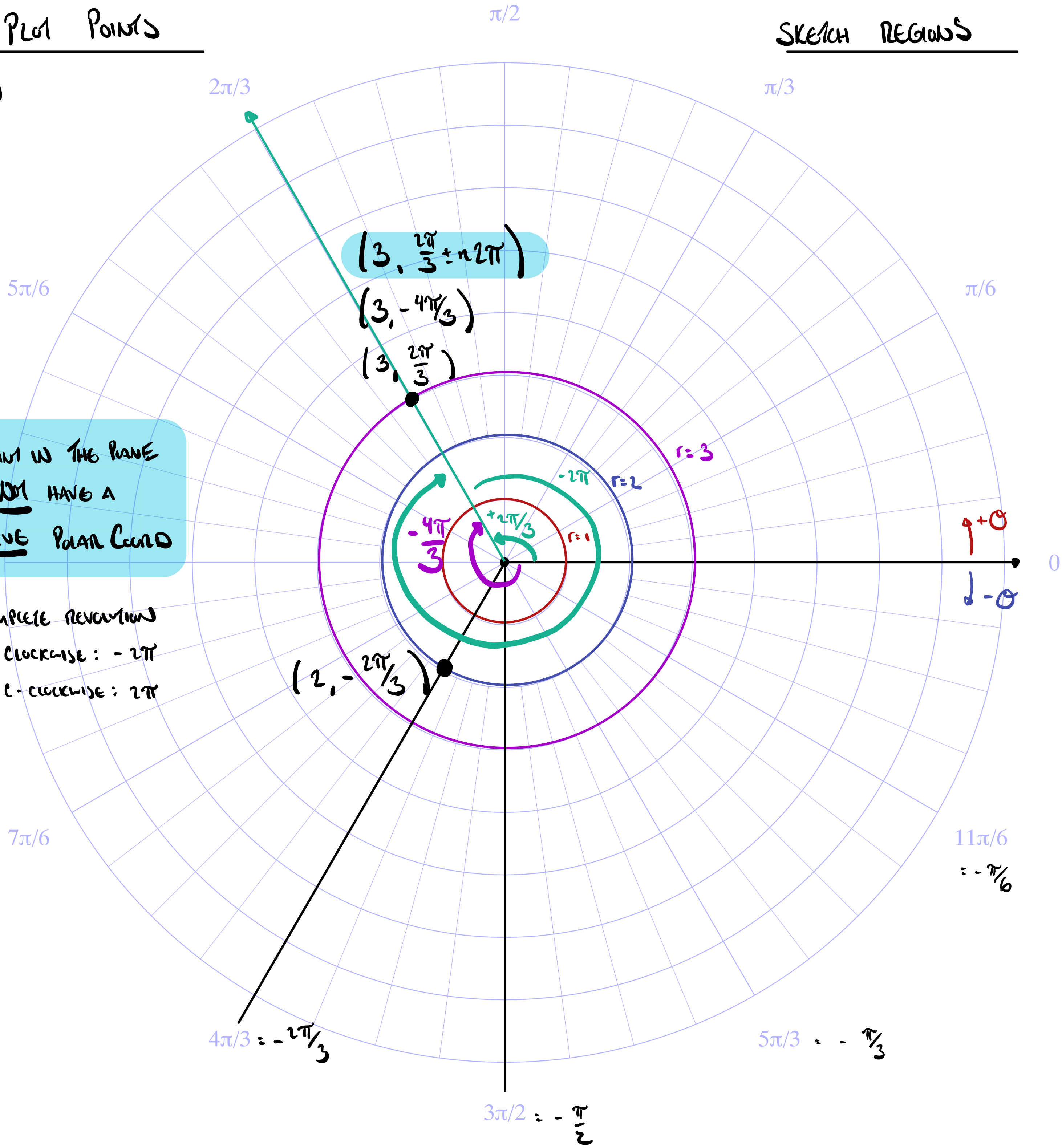
If $r < 0$ → walk backwards distance $|r|$ to P.



Plot Points

(r, θ)

Sketch Regions

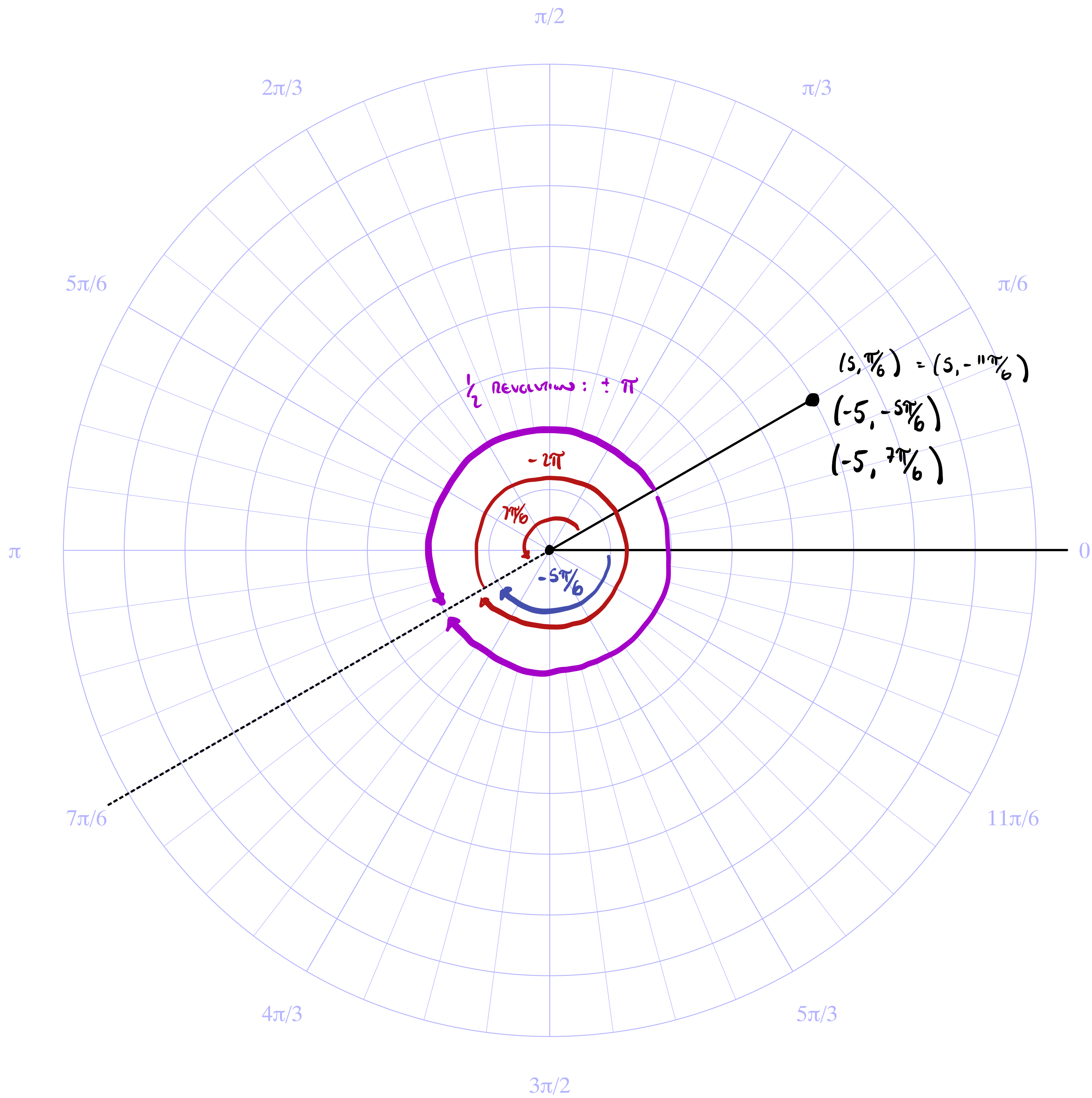


Each point in the plane does **not** have a **unique** polar coord

Complete revolution

Clockwise : -2π

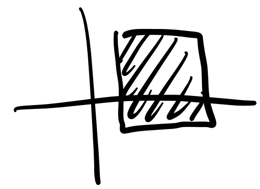
C-clockwise : 2π

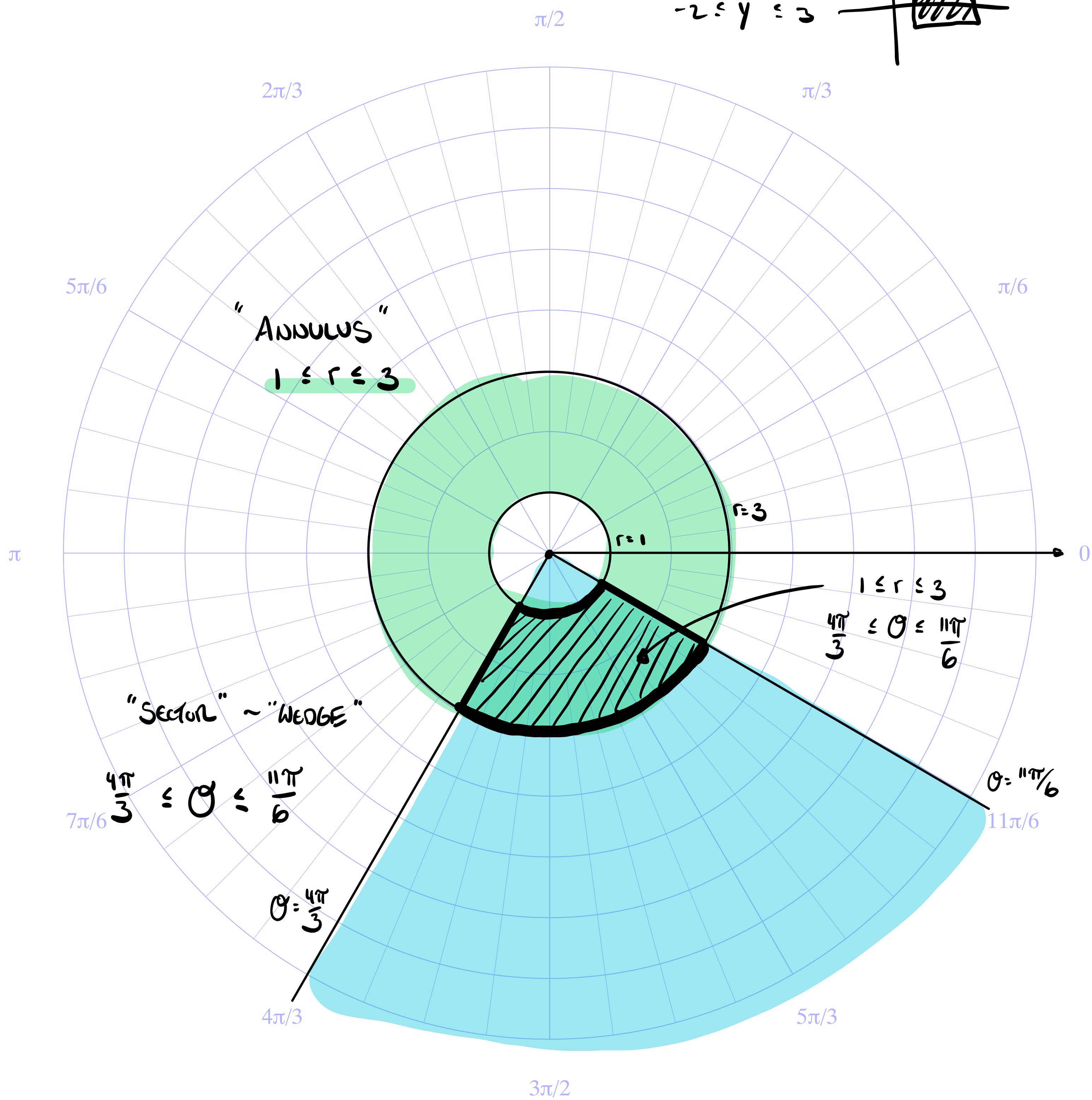

π/2
2π/3
π/3
5π/6
π/6
π
0
7π/6
11π/6
4π/3
5π/3
3π/2
(5, π/6) = (5, -11π/6)
(-5, -5π/6)
(-5, 7π/6)
1/2 revolution : ± π
-2π
7π/6
-5π/6

PLOT REGIONS

$1 \le x \le 2$

$-2 \le y \le 3$

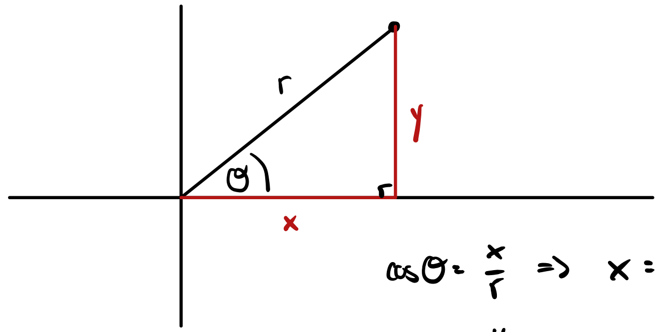




Converting Polar → Cartesian

UNIQUE!

$$x = r\cos\theta \qquad y = r\sin\theta$$



$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

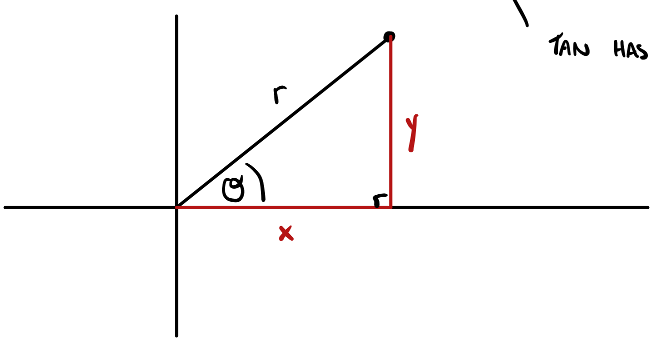
$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$ ✓

Converting Cartesian → Polar

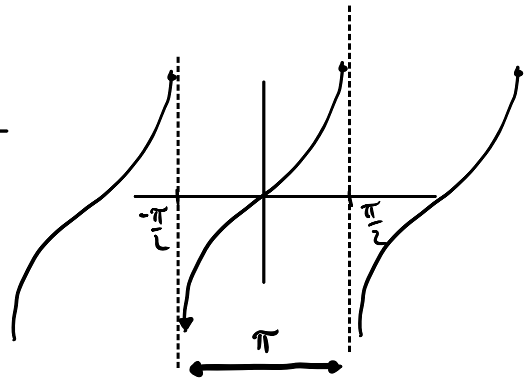
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

$r = \pm\sqrt{x^2+y^2}$

NOT UNIQUE!



TAN HAS PERIOD $\tan(\theta) = \tan(\theta \pm n\pi) = \frac{y}{x}$

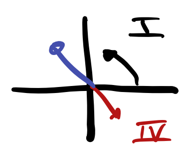


ex. CONVERT POLAR $(6, -5\pi/6)$ TO CARTESIAN: $x = r\cos\theta = (6)\cos(-5\pi/6)$

$= (6)\left(-\sqrt{3}/2\right) = -3\sqrt{3}$

$y = r\sin\theta = (6)\sin(-5\pi/6)$
$(6)(-\tfrac{1}{2}) = -3$

ex. CONVERT CARTESIAN $(2, -2\sqrt{3})$ TO POLAR:



$r^2 = x^2 + y^2 = (2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16$

$r = \pm 4$ $\quad \tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3} = \frac{-\sqrt{3}/2}{1/2} = \frac{\sin\theta}{\cos\theta} \Rightarrow \theta = -\pi/3$

$\pm$ $(4, -\pi/3)$ $+\pi$

$(-4, 2\pi/3)$

$(4, 5\pi/3)$

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

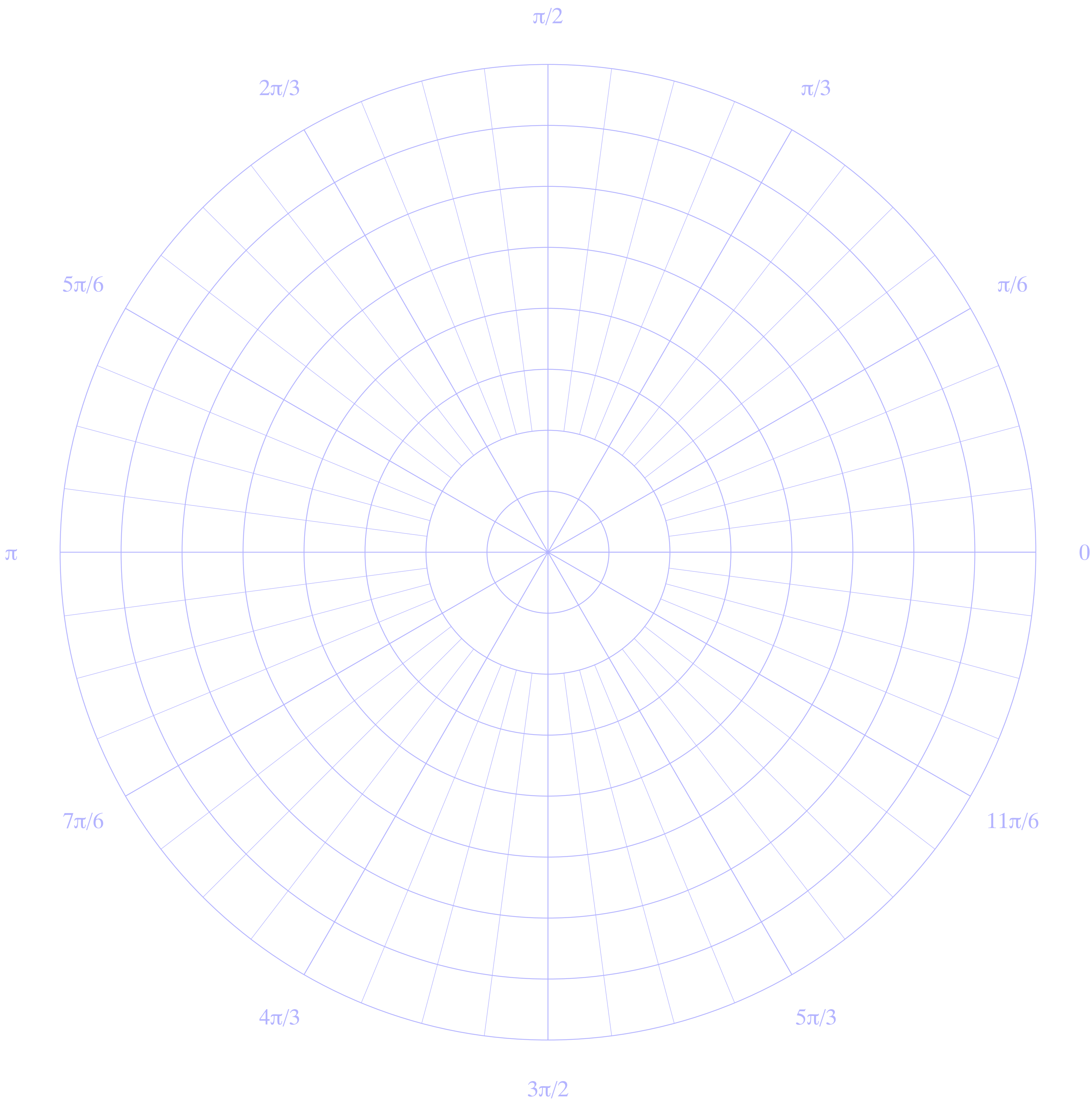
$(-4, \frac{8\pi}{3})$

$(4, \frac{11\pi}{3})$

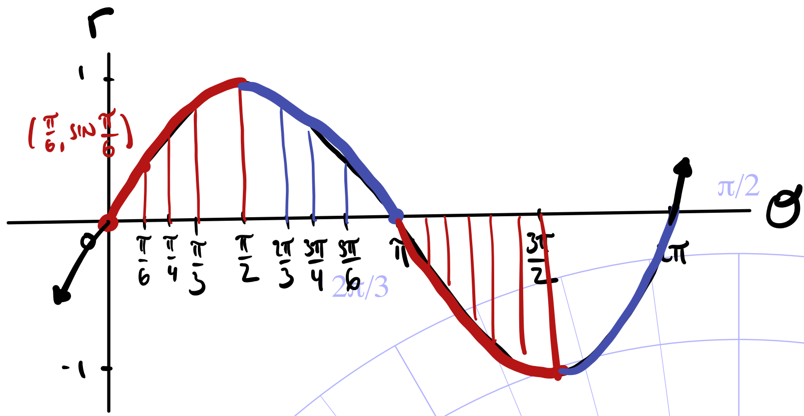
$\vdots$

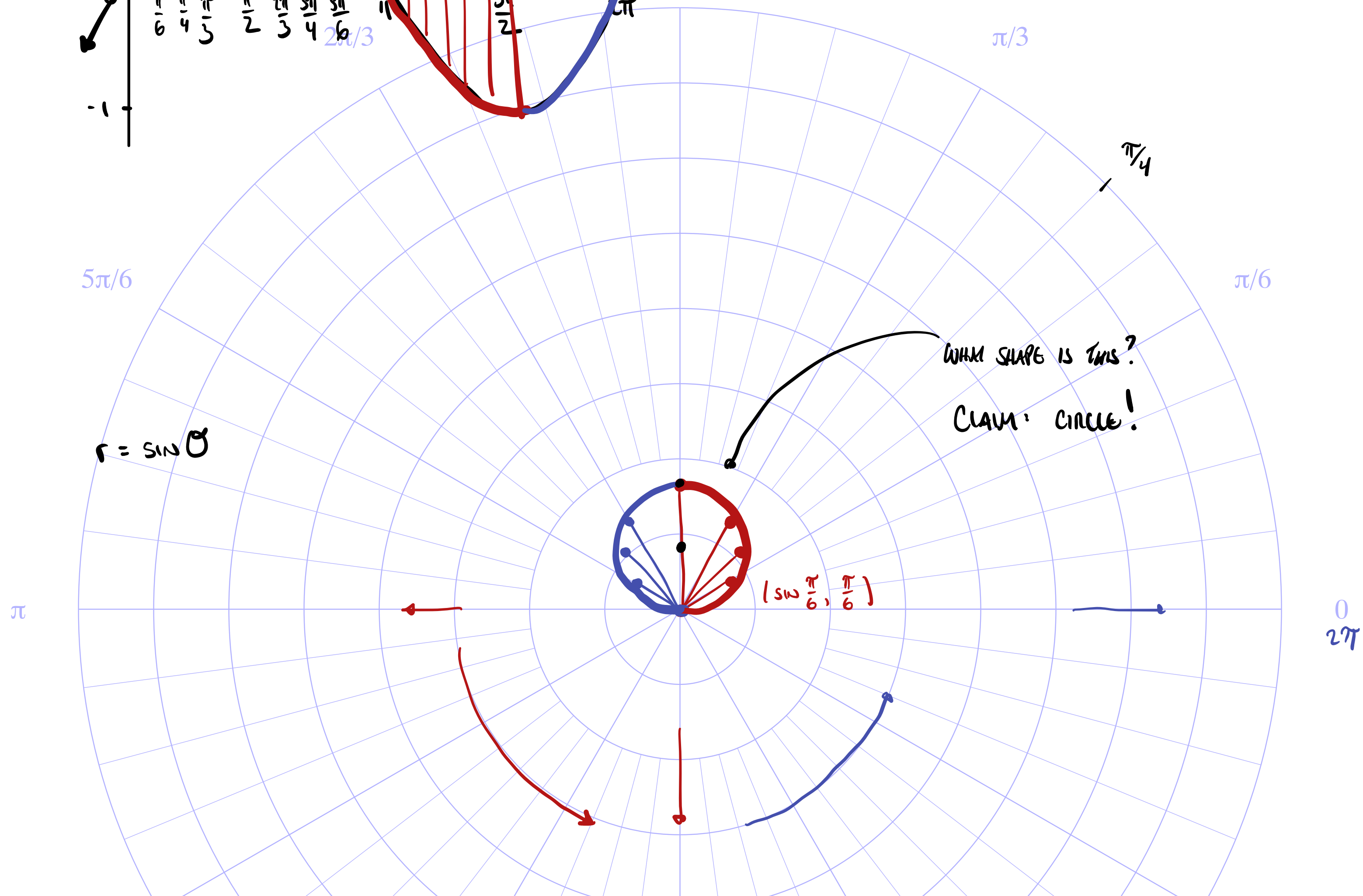
ex. SKETCH THE POLAR GRAPH $r = 2$

ex. SKETCH THE POLAR GRAPH $\theta = 2\pi/3$



ex. Sketch the polar graph $r = \sin\theta$





Convert polar Eq → Cartesian Eq.

$$\boxed{r = \sin\theta} \rightarrow \quad x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$x^2 + y^2 = \boxed{r^2 = r\sin\theta} = y \implies x^2 + y^2 = y$$

$$x^2 + y^2 - y = 0$$

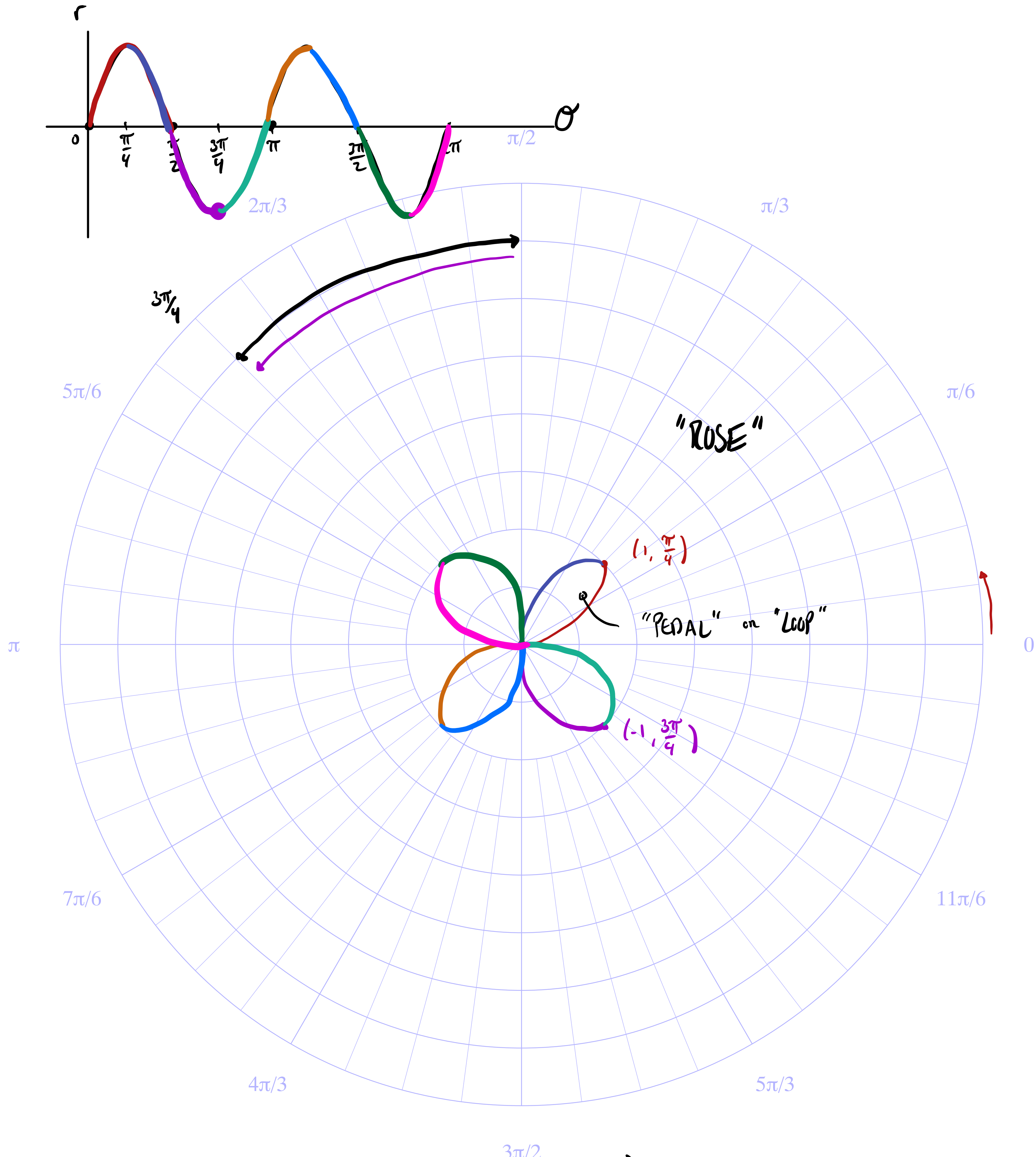
$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Circle: radius $\frac{1}{2}$

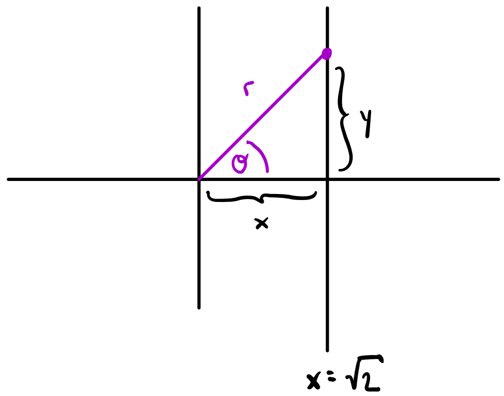
center $\left(0, \frac{1}{2}\right)$

ex. Sketch the polar graph $r = \sin 2\theta$ Period: $\frac{2\pi}{2} = \pi$



Number of pedals on a rose $r = R\sin(n\theta)$ or $r = R\cos(n\theta)$

ex. CONVERT THE CARTESIAN EQ $x = \sqrt{2}$ TO A POLAR EQ.



$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad x^2 + y^2 = r^2$

$x = \sqrt{2} \implies r\cos\theta = \sqrt{2}$

$\implies r = \dfrac{\sqrt{2}}{\cos\theta} \implies \boxed{r = \sqrt{2}\sec\theta}$

Tangents to Polar Curves

POLAR EQ

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

PARAMETRIC EQ'S

PARAMETER θ

Then, using the method for finding slopes of parametric curves (Equation 10.2.1) and the Product Rule, we have

3

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

ex. (a) SKETCH THE CARDIOID CURVE $r = 1 + \cos\theta$

(b) FIND SLOPE OF THE TANGENT LINE WHEN $\theta = 2\pi/3$

(c) FIND ALL POINTS WHERE TANGENT LINE IS HORIZONTAL / VERTICAL.

$$\frac{dy}{dx} = \frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{-\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta}$$

HORIZONTAL: $-\sin^2\theta + \cos\theta + \cos^2\theta = 0 \quad , \quad -\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta \neq 0$

$-(1 - \cos^2\theta) + \cos\theta + \cos^2\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = -1 \implies \theta = \pi$ DENOM $= 0$ TOO!

$\implies \cos\theta = \frac{1}{2} \implies \theta = \pm\frac{\pi}{3}$

VERTICAL: $-\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta = 0 \quad , \quad -\sin^2\theta + \cos\theta + \cos^2\theta \neq 0$