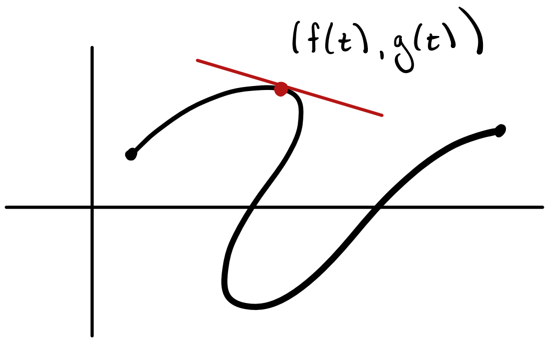
§ 10.2 Calculus with Parametric Curves

- Slope $\frac{dy}{dx}$
- Concavity $\frac{d^2y}{dx^2}$
- Area
- Arclength
- Surface Area

Given parametric Eq's $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ Horizontal position at time t, Vertical position at time t

$a \le t \le b$



Then $\frac{dx}{dt} = f'(t)$ gives horizontal velocity at time t

$\frac{dy}{dt} = g'(t)$ gives vertical velocity at time t

$\Rightarrow$ Slope of the tangent line at point $(f(t), g(t))$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

ratio of vertical velocity over horizontal velocity.

Note: Horizontal tangent when $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$

Vertical tangent when $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$

In the language of vectors:

$\vec{s}(t) = \langle f(t), g(t) \rangle$

$\vec{v}(t) = \langle f'(t), g'(t) \rangle$

9–10 Find an equation of the tangent to the curve at the given point. Then graph the curve and the tangent.

9. $x = t^2 - t, \quad y = t^2 + t + 1; \quad (0, 3)$

10. $x = \sin \pi t, \quad y = t^2 + t; \quad (0, 2)$

17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. $x = t^3 - 3t, \quad y = t^2 - 3$

18. $x = t^3 - 3t, \quad y = t^3 - 3t^2$

19. $x = \cos \theta, \quad y = \cos 3\theta$

20. $x = e^{\sin \theta}, \quad y = e^{\cos \theta}$

ex. PARAMETRIC CURVE C HAS PARAMETRIC EQ'S

$$\begin{cases} x = \sin(\pi t) \\ y = t^3 - t^2 \end{cases}, \quad -2 \le t \le 2$$

a. FIND SLOPE OF TANGENT AT $(0, -2)$

$$\frac{dy}{dx} = \frac{3t^2 - 2t}{\pi \cos(\pi t)} \longrightarrow t = -1 \ \ldots$$

b. SHOW THAT THE CURVE HAS 2 TANGENTS AT THE ORIGIN.

$$\frac{dy}{dx} = \frac{3t^2 - 2t}{\pi \cos(\pi t)}$$

$t = 0 \Rightarrow \frac{dy}{dx} = 0$

$t = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\pi}$

Second Derivative:

RECALL: $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{d}{dx}[y] = \frac{\frac{d}{dt}[y]}{\frac{dx}{dt}}$$

THUS $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

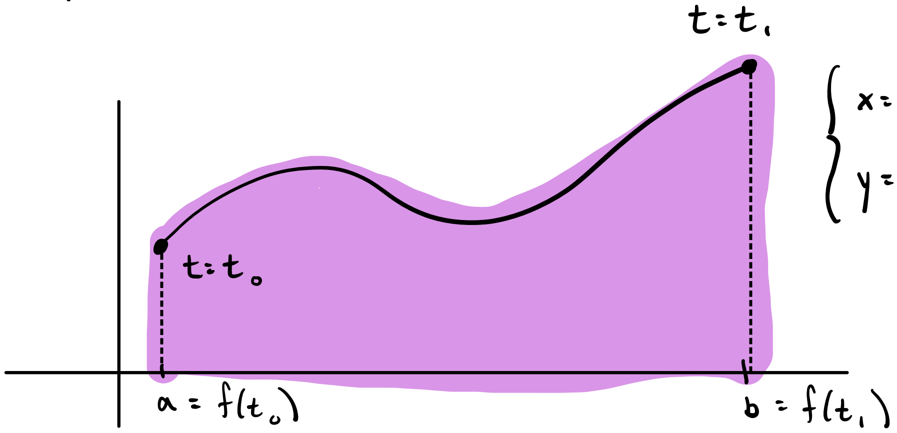
RATIO OF CHANGE IN SLOPE WRT TIME OVER CHANGE IN HORIZONTAL POSITION WRT TIME

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}}$$

ex. FIND $\frac{d^2y}{dx^2}$ WHEN $x = t^2$
$y = 2t^3 - 3t^2$

FOR WHAT VALUES OF t
IS THE PARAM. CURVE
CONCAVE UP / DOWN ?

AREAS



$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

$$\text{AREA} = \int_a^b y\,dx = \int_{t_0}^{t_1} g(t)\,f'(t)\,dt$$

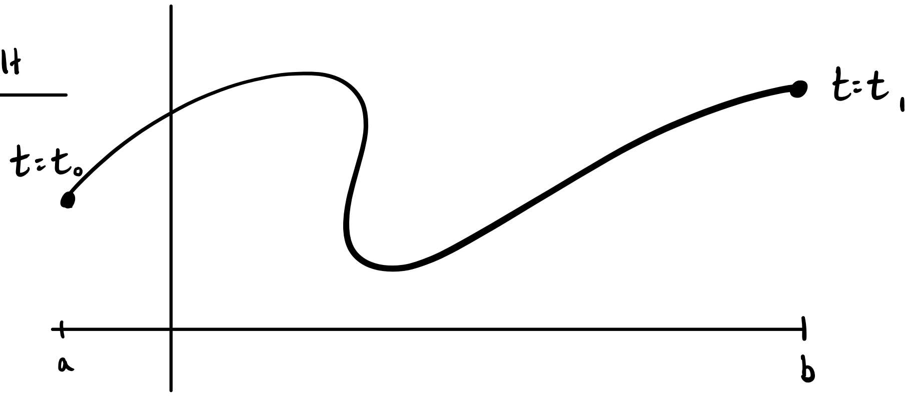
NOTE THAT $\underset{\text{LEFT}}{f(t_0)} \le \underset{\text{RIGHT}}{f(t_1)}$

(NOT NECESSARILY $t_0 \le t_1$)

32. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

33. Find the area enclosed by the x-axis and the curve $x = t^3 + 1$, $y = 2t - t^2$.

ARC LENGTH



RECALL ARC LENGTH $L = \displaystyle\int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\; dx$

$$= \int_{t_0}^{t_1} \sqrt{\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2 + 1}\;\; \frac{dx}{dt}\, dt$$

$$\boxed{L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt}$$

NOTE: $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$

$$\underbrace{\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{\text{SPEED OF PARTICLE AT TIME } t}$$

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t + e^{-t}, \quad y = t - e^{-t}, \quad 0 \leqslant t \leqslant 2$

38. $x = t^2 - t, \quad y = t^4, \quad 1 \leqslant t \leqslant 4$

39. $x = t - 2\sin t, \quad y = 1 - 2\cos t, \quad 0 \leqslant t \leqslant 4\pi$

40. $x = t + \sqrt{t}, \quad y = t - \sqrt{t}, \quad 0 \leqslant t \leqslant 1$

41–44 Find the exact length of the curve.

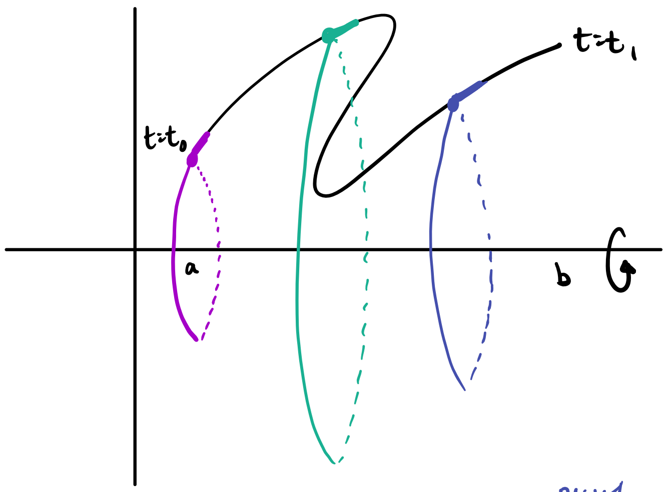
41. $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$

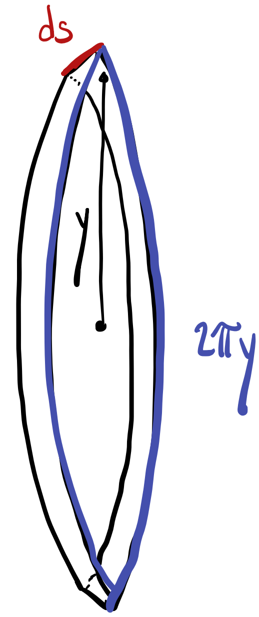
42. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 2$

43. $x = t\sin t, \quad y = t\cos t, \quad 0 \leqslant t \leqslant 1$

44. $x = 3\cos t - \cos 3t, \quad y = 3\sin t - \sin 3t, \quad 0 \leqslant t \leqslant \pi$

SURFACE AREA





RIGHT

$$S = \int_a^b 2\pi y \, ds = \int_{t_0}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

LEFT

TOP

$$S = \int_c^d 2\pi x \, ds \qquad \int_{t_0}^{t_1} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

BOTTOM

61–63 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

61. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

62. $x = 2t^2 + 1/t, \quad y = 8\sqrt{t}, \quad 1 \leqslant t \leqslant 3$

63. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

WA §10.2 #6

FIND EQUATIONS OF THE TANGENTS TO THE CURVE

$$\begin{cases} x = 9t^2 + 3 \\ y = 6t^3 + 3 \end{cases}$$

THAT PASS THROUGH THE POINT $(12, 9)$.

FOR EVERY $t \in \mathbb{R}$, $(x(t), y(t))$ IS A POINT ON THE PARAMETRIC CURVE, AND $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$ GIVES THE SLOPE OF THE TANGENT LINE TO THE CURVE AT THAT POINT.

$\therefore$ FOR EVERY $t \in \mathbb{R}$, $y - y(t) = \dfrac{y'(t)}{x'(t)}(x - x(t))$ [POINT-SLOPE EQ.]

GIVES A TANGENT LINE TO THE PARAMETRIC CURVE

$$y - (6t^3 + 3) = \frac{18t^2}{18t}(x - (9t^2 + 3))$$

$$(*) \quad y - 6t^3 - 3 = xt - 9t^3 - 3t$$

THIS LINE PASSES THROUGH $(12, 9)$ IF $x = 12$, $y = 9$ SATISFIES THIS EQ.

$$9 - 6t^3 - 3 = 12t - 9t^3 - 3t$$

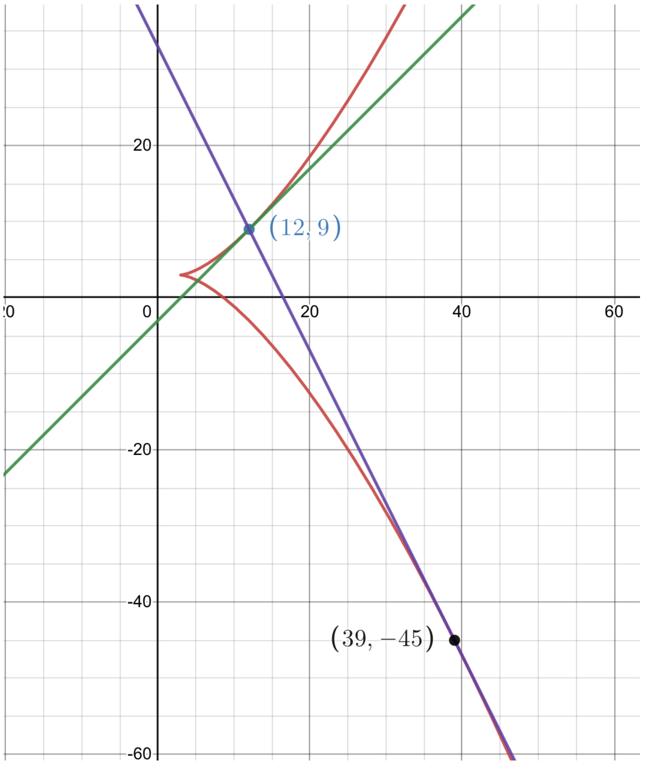
$$3t^3 - 9t + 6 = 0$$

THERE ARE 2 SOLUTIONS: $t = 1$, $t = -2$ (THIS IS NOT OBVIOUS, I FOUND SOLUTIONS BY "GUESS & CHECK")

EQ $(*)$ FOR THESE VALUES OF t BECOMES:

$t = 1$: $y = x - 3$ (TANGENT @ $\underset{(x(1), y(1))}{(12, 9)}$, PASSES THROUGH $(12, 9)$)

$t = -2$: $y = -2x + 33$ (TANGENT @ $\underset{(x(-2), y(-2))}{(39, -45)}$, PASSES THROUGH $(12, 9)$)



https://www.desmos.com/calculator/llmy2ydrgf

PLAY WITH THE SLIDER FOR s!