Find the critical numbers of the function. (Enter your answers as a comma-separated list. Use n to denote any arbitrary integer values. If an answer does not exist, enter DNE.)

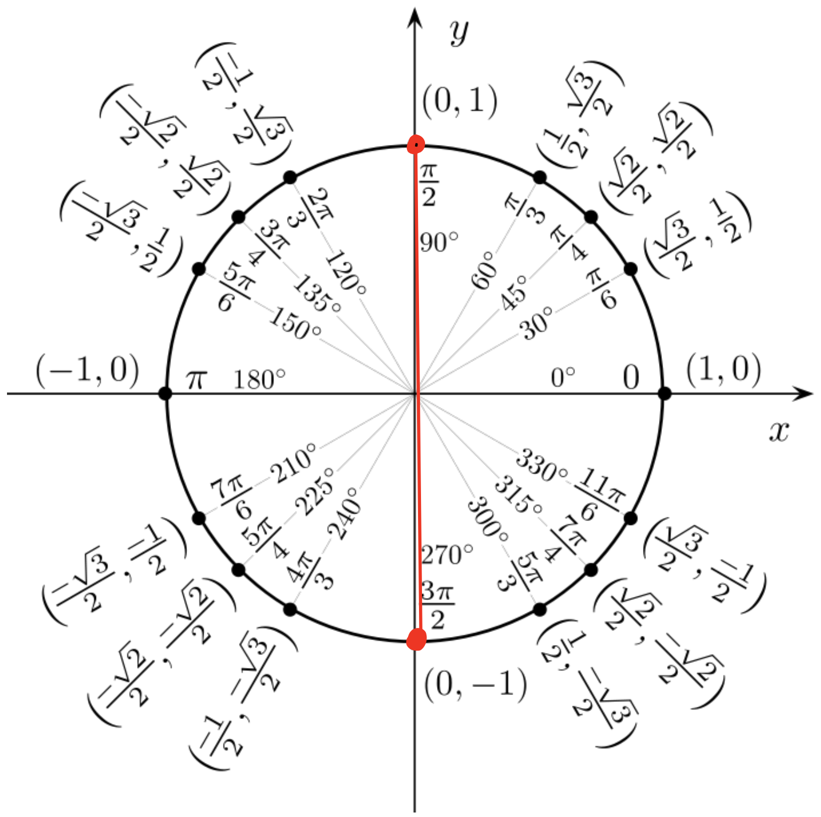
$g(\theta) = 36\theta - 9\tan(\theta)$

$\theta =$ $\boxed{2\pi n + \frac{\pi}{3}, 2\pi n + \frac{2\pi}{3}, 2\pi n + \frac{4\pi}{3}, 2\pi n + \frac{5\pi}{3}}$

$$g(\theta) = 16\theta - 4\tan\theta = 16\theta - \frac{4\sin\theta}{\cos\theta}$$

$$\text{Dom}(g) = \{\theta \in \mathbb{R} \mid \cos\theta \neq 0\} = \{\theta \in \mathbb{R} \mid \theta \neq \frac{\pi}{2} + n2\pi, \theta \neq \frac{3\pi}{2} + n2\pi\}$$

$x = 0$



$x = \cos\theta$

$y = \sin\theta$

FIND CRIT. #'S : g' EITHER 0 OR UND.

g' UNDEFINED WHEN

$\cos^2\theta = 0$

$\cos\theta = 0$

⊗ NOT CRIT. PTS NOT IN DOMAIN

$$g'(\theta) = 16 - 4\sec^2\theta = 16 - \frac{4}{\cos^2\theta}$$

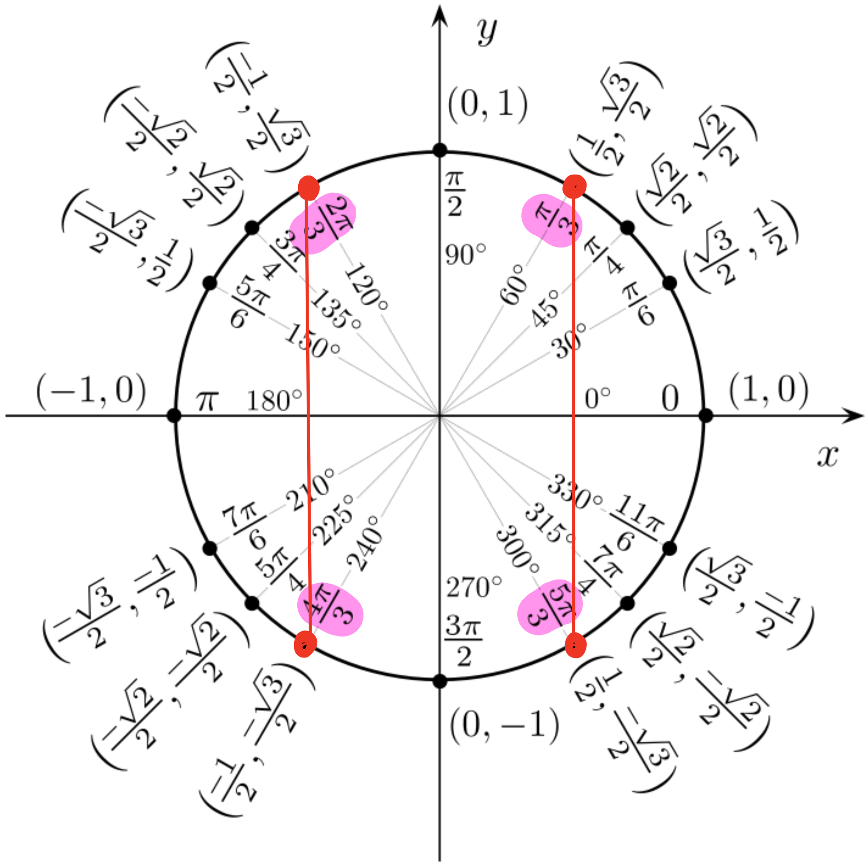
$$g'(\theta) = 0 \Rightarrow 16 = \frac{4}{\cos^2\theta}$$

$$16\cos^2\theta = 4$$

$\cos^2\theta = \frac{4}{16} = \frac{1}{4}$

$\cos\theta = \pm\frac{1}{2}$

$x = \pm\frac{1}{2}$



crit. #'s :

$\frac{\pi}{3} + n2\pi$

$\frac{2\pi}{3} + n2\pi$

$\frac{4\pi}{3} + n2\pi$

$\frac{5\pi}{3} + n2\pi$

$\frac{4\pi}{3} = \frac{\pi}{3} + \pi$

$\frac{5\pi}{3} = \frac{2\pi}{3} + \pi$

$\frac{\pi}{3} + n\pi$

$\frac{2\pi}{3} + n\pi$

Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = x + \frac{16}{x}, \quad [0.2, 16]$$

absolute minimum value 8

absolute maximum value 80.2

$f(x) = x + \frac{4}{x}$, $[.2, 8]$ — f IS CONT. ON $[.2, 8]$ ✓

CONT. EVERYWHERE EXCEPT 0.

ABS MAX OCCUR @ EITHER LOCAL MAX/MINS (WHICH HAPPEN AT CRIT. #'s) OR @ ENDPOINTS OF THE CLOSED INTERVAL.

POSSIBILITIES: CRIT. #'s, ENDPOINTS

$f(x) = x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = 0$ UNDEFINED AT $x = 0 \notin \mathrm{DOM}(f)$

$1 = \frac{4}{x^2} \Rightarrow x^2 = 4$

$x = \pm 2$

POINT	$f(x)$
$x = .2$	$.2 + \frac{4}{.2} = 20.2$
$x = 2$	$2 + \frac{4}{2} = 4$
$x = 8$	$8 + \frac{4}{8} = 8.5$

ABS MAX VALUE 20.2 @ $x = .2$

ABS MIN VALUE 4 @ $x = 2$

§3.4 Horizontal Asymptotes, Limits at Infinity

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_n \neq 0$

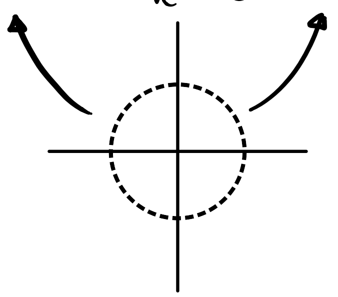
be an n^{th} degree polynomial with lead coefficient a_n.

Then, when $|x|$ is very large

$$f(x) \approx a_n x^n$$ dominated by first term.

And the graph $y = f(x)$ has end behavior...

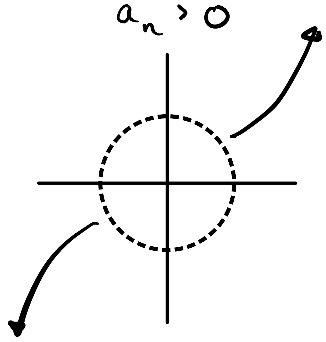
n even
$a_n > 0$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

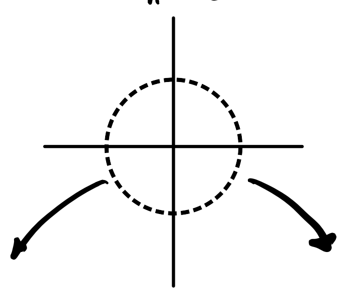
n odd
$a_n > 0$



$\lim_{x \to \infty} f(x) = \infty$

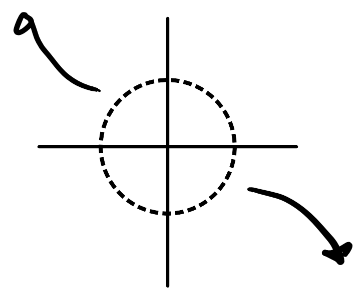
$\lim_{x \to -\infty} f(x) = -\infty$

n even
$a_n < 0$



$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = -\infty$



$\lim_{x \to \infty} f(x) = -\infty$

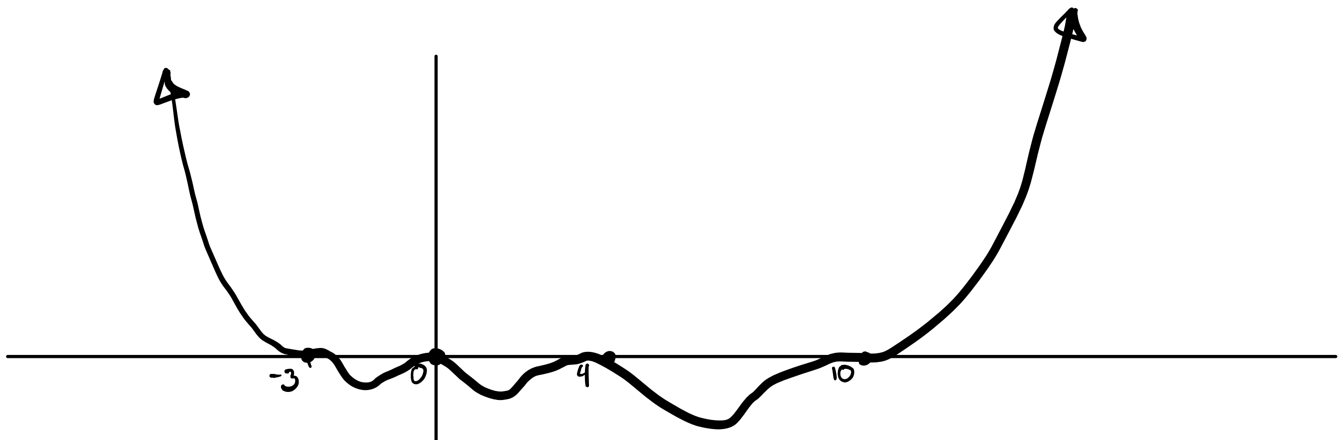
$\lim_{x \to -\infty} f(x) = \infty$

ex SKETCH $y = x^2(x+3)^3(x-4)^4(x-10)^5$ ← DEGREE $n = 14$

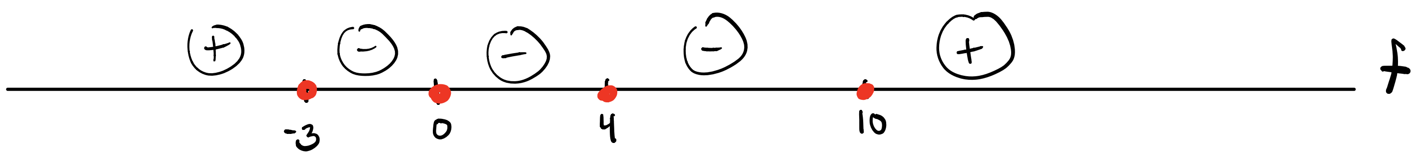
$= x^2(x^3+\dots)(x^4+\dots)(x^5+\dots)$ LEAD COEFF $a_n = 1 > 0$

$= x^{14} + \dots$

x-INT: $y = 0 \Rightarrow x = -3, 0, 4, 10$



At zeros of factors with odd exponents, the graph crosses the x-axis. At zeros of factors with even exponents, the graph bounces off the x-axis.



x^2	+	+	+	+	+
$(x+3)^3$	−	+	+	+	+
$(x-4)^4$	+	+	+	+	+
$(x-10)^5$	−	−	−	−	+

THM.

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0}$$

DEGREE n POLYNOMIAL WITH LEAD COEFFICIENT a_n

DEGREE m POLYNOMIAL WITH LEAD COEFFICIENT b_m

$$= \begin{cases} \pm\infty & \text{IF } n > m \quad \left(+\infty \text{ IF } \frac{a_n}{b_m} > 0,\ -\infty \text{ IF } \frac{a_n}{b_m} < 0\right) \\ \frac{a_n}{b_m} & \text{IF } n = m \\ 0 & \text{IF } n < m \end{cases}$$

MUCH BIGGER THAN

IDEA: WHEN $|x|$ IS LARGE, $x^n \gg x^{n-k}$, $k = 1, 2, 3 \dots$

AND SO $$\frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0} \approx \frac{a_n x^n}{b_m x^m}.$$

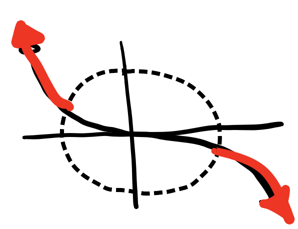
THUS

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0} = \frac{a_n}{b_m} \lim_{x\to\infty} x^{n-m}.$$

ex. DESCRIBE $\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$

WHEN (a) $f(x) = \dfrac{x^2 + 2x - 7}{2x^3 - 5x^2 + 3} \approx \dfrac{x^2}{2x^3}$ WHEN $|x|$ IS LARGE

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{2x^3} = \frac{1}{2}\lim_{x\to\pm\infty}\frac{1}{x} = 0$$



(b) $f(x) = \dfrac{-4x^5 + 3x + 1}{5x^2 + x - 1} \approx \dfrac{-4x^5}{5x^2} = -\dfrac{4}{5}x^3$ WHEN $|x|$ IS LARGE

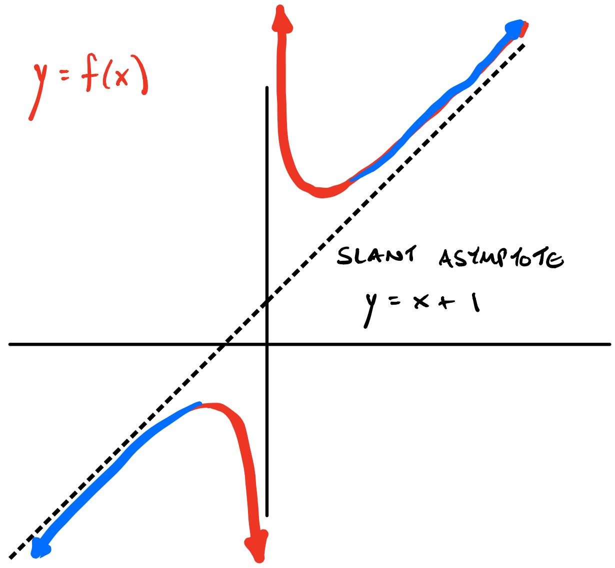
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -\frac{4}{5}x^3 = -\infty, \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -\frac{4}{5}x^3 = \infty$$

SLANT ASYMPTOTES

LET $f(x) = \underbrace{x + 1}_{} + \underbrace{\frac{1}{x}}_{}$

WHEN $|x|$ IS LARGE, $\frac{1}{x} \approx 0$

$\Rightarrow f(x) \approx x + 1$



IN GENERAL, $y = f(x)$ HAS SLANT ASYMPTOTE $y = mx + b$

IF $\lim_{x \to \infty} \left| f(x) - (mx + b) \right| = 0$, OR

$\lim_{x \to -\infty} \left| f(x) - (mx + b) \right| = 0$.

EQUIVALENTLY, ... IF

$$\lim_{x \to \pm\infty} \frac{f(x)}{mx + b} = 1$$

$$\left(\begin{aligned} \frac{f(x)}{mx+b} &\approx 1 \\ f(x) &\approx mx + b \end{aligned} \right)$$

EX. FIND AN EQUATION OF THE SLANT ASYMPTOTE FOR

$$y = \frac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x} \qquad \lim_{x \to \infty}$$

WRONG:

$$y = \frac{4x^3}{x^2 - 3x} - \frac{10x^2}{x^2 - 3x} - \frac{11x}{x^2 - 3x} + \frac{1}{x^2 - 3x}$$

$$y \approx \frac{4x^3}{x^2} - \frac{10x^3}{x^2} - \underbrace{\frac{11x}{x^2} + \frac{1}{x^2}}_{0}$$

$$y \approx 4x - 10 \qquad \text{FOR LARGE } |x|$$

CORRECT: LONG DIVISION:

$$\begin{array}{r} 4x + 2 \\ x^2 - 3x \,\big)\, \overline{4x^3 - 10x^2 - 11x + 1} \\ -(4x^3 - 12x^2) \\ \hline 2x^2 - 11x + 1 \\ -(2x^2 - 6x) \\ \hline -5x + 1 \end{array} \quad \text{REMAINDER}$$

SLANT ASYMPTOTE

↓

$$f(x) = 4x + 2 + \underbrace{\frac{-5x + 1}{x^2 - 3x}}_{\to 0 \text{ AS } x \to \pm\infty}$$

§3.5 Summary of Curve Sketching

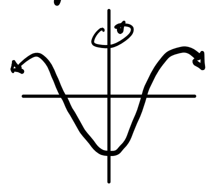
Things to keep in mind:

A. Domain

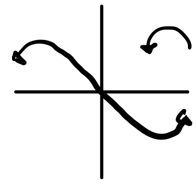
B. Intercepts
- x-int: set $y=0$, solve for x
- y-int: set $x=0$, solve for y

C. Symmetry

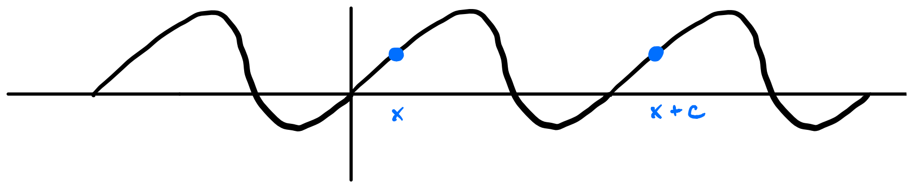
Even: $f(-x) = f(x)$



Odd: $f(-x) = -f(x)$



Periodic: $f(x+c) = f(x)$



D. Asymptotes
- Vertical: $x = c$, any number
- Horizontal: $y = c$, at most 2
- Slant: $y = mx + b$, at most 2, usually 0 or 1

E, F. Critical #'s

↓

Intervals of incr / decr. $(f' > 0 \,/\, f' < 0)$

↓

Local max/mins (x-coord.), local max/min values (y-coord.)

G. Intervals of concave up / concave down $(f'' > 0,\ f'' < 0)$

↓

Points of Inflection

H. Sketch!

4. $y = x^4 - 8x^2 + 8$

DOMAIN : $\mathbb{R}$

INTERCEPTS : y-INT : $y = 0^4 - 8(0)^2 + 8 = 8$

x-INT : LET $w = x^2 \rightarrow y = w^2 - 8w + 8 = 0$

$$w = \frac{8 \pm \sqrt{64 - 4(1)(8)}}{2(1)} = \frac{8 \pm \sqrt{32}}{2}$$

$$w = 4 \pm 2\sqrt{2}$$

$$x^2 = 4 \pm 2\sqrt{2} \rightarrow x = \pm\sqrt{4 \pm 2\sqrt{2}}$$

$$x = -\sqrt{4-2\sqrt{2}},\ -\sqrt{4+2\sqrt{2}},\ \sqrt{4-2\sqrt{2}},\ \sqrt{4+2\sqrt{2}}$$

SYMMETRY : EVEN SYMMETRY

$$\lim_{x \to \infty} x^4 - 8x^2 + 8 \approx \lim_{x \to \infty} x^4 = \infty$$

$$\lim_{x \to -\infty} x^4 - 8x^2 + 8 \approx \lim_{x \to -\infty} x^4 = \infty$$

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$$

$f'(x) = 0$ WHEN $x = -2, 0, 2$

1st DERIV. TEST.

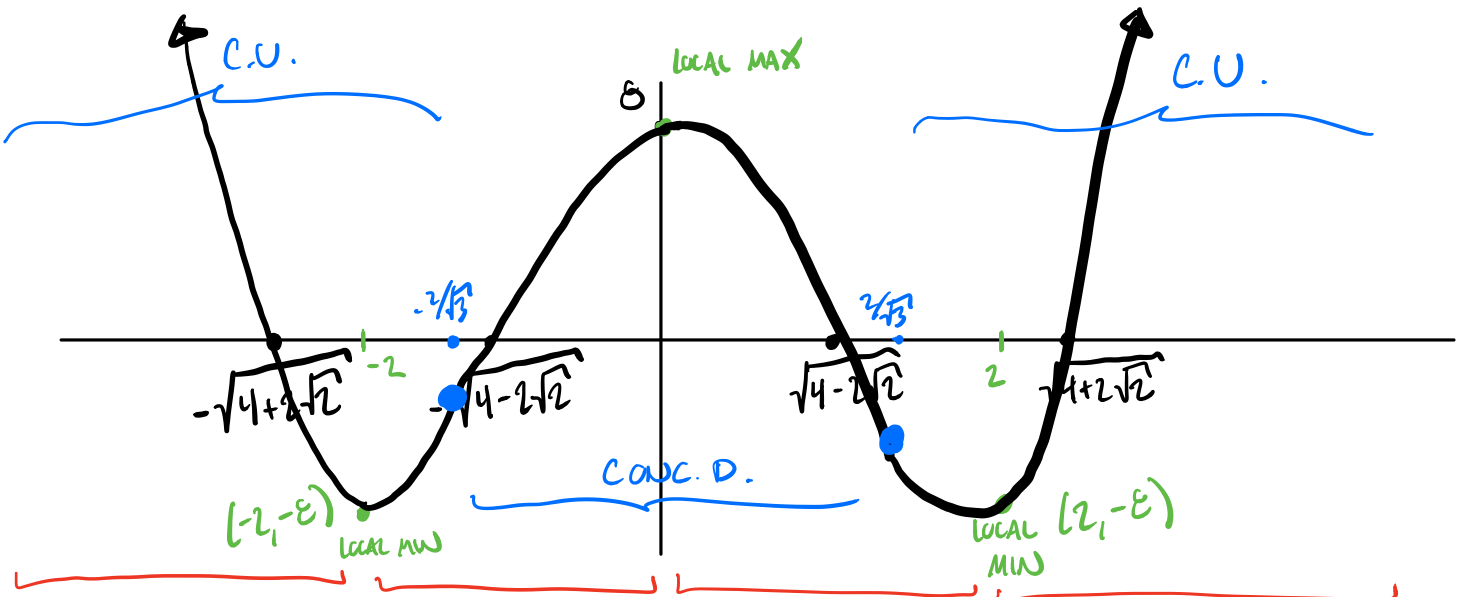
		-2		0		2	
f	$\searrow$	LOCAL MIN	$\nearrow$	LOCAL MAX	$\searrow$	LOCAL MIN	$\nearrow$
f'	$-$	0	$+$	0	$-$	0	$+$
$(x+2)$	$-$		$+$		$+$		$+$
$4x$	$-$		$-$		$+$		$+$
$(x-2)$	$-$		$-$		$-$		$+$

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left[4x^3 - 16x\right]$$

$$f''(x) = 12x^2 - 16 = 4(3x^2-4) = 0$$

$$3x^2 - 4 = 0 \qquad 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\frac{2}{\sqrt{3}}$$

		$-\frac{2}{\sqrt{3}}$		$\frac{2}{\sqrt{3}}$	
f	CU		CD		CU
f''	$+$		$-$		$+$
$4(3x^2-4)$	$+$		$-$		$+$



$x = -\sqrt{4-2\sqrt{2}}, -\sqrt{4+2\sqrt{2}}, \sqrt{4-2\sqrt{2}}, \sqrt{4+2\sqrt{2}}$

$f(2) = (2)^4 - 8(2)^2 + 8 = 16 - 32 + 8 = -8$

CHECK: