Recitation Calculus I (Adamski)

Tuesday, December 8, 2020

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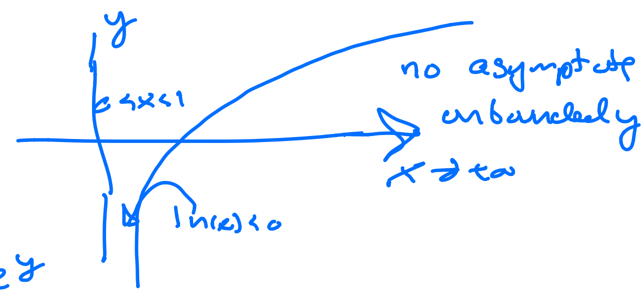
① c) $\lim_{x \to +\infty} \frac{\sin(x)}{\ln(x)} = \frac{-1 \le \sin(x) \le 1}{+\infty} = 0$

$y = \ln(x)$

$|\sin(x)| \le 1$

$\lim_{x \to +\infty} \ln(x) = \underline{+\infty}$



$\ln(e^y) = y$

For any $y > 0$, set $x = e^y$

then $\underline{\ln(x)} = \ln(e^y) = \underline{y}$

$$\frac{-1}{|\ln(x)|} \le \frac{\sin(x)}{\ln(x)} \le \frac{1}{|\ln(x)|}$$

Since $\lim_{x \to +\infty} \frac{1}{|\ln(x)|} = 0$, $\therefore$ By Squeeze Theorem $\lim_{x \to +\infty} \frac{\sin(x)}{\ln(x)} = 0$ too.

b) $\lim_{\substack{x \to 0^+ \\ x > 0}} \left(\underline{\ln\left(\frac{1}{x}\right)} - \ln\left(\frac{1}{\sin(4x)}\right) \right)$ "$+\infty - +\infty$" (Hard)

$\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$ ← first step

$$= \lim_{\substack{x \to 0 \\ x > 0}} \ln\left(\frac{1/x}{\frac{1}{\sin(4x)}} \right) = \lim_{\substack{x \to 0 \\ x > 0}} \ln\left(\frac{\sin(4x)}{x}\right)$$

$$= \ln\left(\lim_{\substack{x \to 0 \\ x > 0}} \frac{\sin(4x)}{x} \right) = \ln(4)$$ because $\ln(x)$ is $\underline{\text{continuous}}$.

If $\lim_{x\to x_0} g(x) = y_0$ and f is continuous at y_0

then $\lim_{x\to x_0} f(g(x)) = f\left(\lim_{x\to x_0} g(x)\right)$.

$$\lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} \frac{\sin(4x)}{4x}\cdot\frac{4x}{x}$$

$(x\neq 0)$

$$= \lim_{\substack{x\to 0\\ \theta=4x}} \frac{\sin(\theta)}{\theta} \lim_{x\to 0} 4 = 1(4) = 4$$

(Hard)

"$\infty-\infty$"

a) $\lim_{x\to+\infty} \sqrt{x^2+2x} - \sqrt{x^2-x}$

$A, B \geq 0$

rationalizing

$$\boxed{(\sqrt{A}-\sqrt{B})} = (\sqrt{A}-\sqrt{B})\frac{(\sqrt{A}+\sqrt{B})}{\sqrt{A}+\sqrt{B}} = \frac{A-B}{\sqrt{A}+\sqrt{B}}$$

$$\sqrt{x^2+2x}-\sqrt{x^2-x} = \frac{(\sqrt{x^2+2x}-\sqrt{x^2-x})(\sqrt{x^2+2x}+\sqrt{x^2-x})}{(\sqrt{x^2+2x}+\sqrt{x^2-x})}$$

$$= \frac{(x^2+2x)-(x^2-x)}{\sqrt{x^2+2x}+\sqrt{x^2-x}} = \frac{3x}{\sqrt{x^2+2x}+\sqrt{x^2-x}}$$

$\frac{\infty}{\infty}$

$$= \lim_{x\to+\infty} \frac{3x}{\sqrt{x^2+2x}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{3}{\sqrt{1+2/x}+\sqrt{1-1/x}} = \frac{3}{2}$$

$$\frac{1}{x}\sqrt{x^2+2x} = \sqrt{\frac{1}{x^2}}\sqrt{x^2+2x} = \sqrt{\frac{1}{x^2}(x^2+2x)} = \sqrt{1+\frac{2}{x}} \to \sqrt{1}$$

$$\frac{1}{x}\sqrt{x^2-x} = \sqrt{\frac{1}{x^2}}\sqrt{x^2-x} = \sqrt{\frac{1}{x^2}(x^2-x)} = \sqrt{1-1/x} \to \sqrt{1}$$

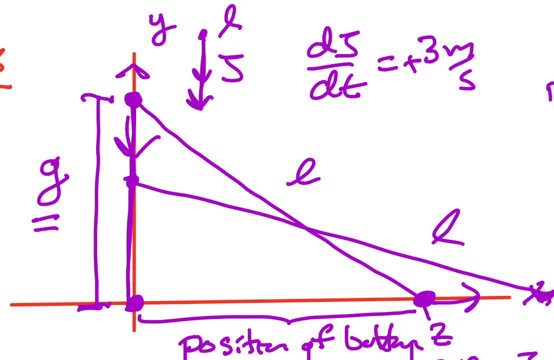
$$\boxed{\lim_{x\to+\infty} \sqrt{x^2+2x}-\sqrt{x^2-x} = \frac{3}{2}}$$

Calculator check?

$x = 1000$

$\sqrt{1000^2+2000} - \sqrt{1000^2-1000} \simeq 1.49962\ldots$

Ex



$\frac{dy}{dt} = +3 \frac{m}{s}$

A bar of length l is attached to the x-axis and y-axis at each end.

$\frac{d}{dt} l = 0$

The top of the bar is sliding down at a rate of 3 m/s when the y-coord is $\frac{1}{2} l$. At what rate is the bottom sliding away away from the origin?

Asked-for $\frac{dz}{dt} = ?$ Given rate $\frac{dy}{dt} = -3 \frac{m}{s}$

Find a relation between z and y

Pythagorean Theorem $y^2 + z^2 = l^2$ (relation)

Differentiate with respect to time.

$\frac{d}{dt} l = 0$; $\frac{d}{dt} l^2 = 0$; $\frac{d}{dt} y^2 = 2y \frac{dy}{dt}$

$\frac{d}{dt} z^2 = 2z \frac{dz}{dt}$

$2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0$

$2\left(\frac{l}{2}\right)(-3) + 2\left(\frac{\sqrt{3}}{2} l\right) \frac{dz}{dt} = 0$

$y^2 + z^2 = l^2$

$\left(\frac{l}{2}\right)^2 + z^2 = l^2$

$z^2 = l^2 - \frac{l^2}{4} = \frac{3}{4} l^2$

$z = \frac{\sqrt{3}}{2} l$

$\sqrt{3} l \frac{dz}{dt} = 3l$; $\frac{dz}{dt} = \frac{3l}{\sqrt{3} l} = \sqrt{3} \frac{m}{s}$.

Ex Let $f(x) = x^3 + bx + a$ where $a, b > 0$.

(a and b are "parameters".)

Show that f has exacty one point $r \in \mathbb{R}$

where $f(r) = 0$. (Show f has only one root.)

Intermediate Value Theorem (IVT) r exists
to show there is a point r.
(Never say what r is!)

Mean Value Theorem [MVT] r is unique.
to: there is at most one root.

(Easier) Suppose there are two roots r_1 and r_2
so that $f(r_1) = 0$ and $f(r_2) = 0$.
f is a polynomial (hence differentiable), so
we can apply MVT to f on interval $[r_1, r_2]$.

Assert existence of point c)

There is a number c in (r_1, r_2) such that $f'(c) = \dfrac{f(r_2) - f(r_1)}{r_2 - r_1}$.

$$f(x) = x^3 + bx + a$$

$$f'(x) = 3x^2 + b \ ; \qquad 3c^2 + b = \frac{0 - 0}{r_2 - r_1} = 0$$

Impossible b/c $3c^2 + b \geq b > 0$. $\therefore$ There cannot be two roots.

Since $\lim\limits_{x \to +\infty} x^3 = +\infty$, $\lim\limits_{x \to +\infty} \underbrace{x^3 + bx + a}_{f(x)} = +\infty$

Since x^3 the dominant term.

$\therefore$ There exists a point x_2 when $\underline{f(x_2) > 0}$

Similarly $\lim_{x \to -\infty} x^3 + bx + a = -\infty$,

so there exists a point x_1 where $f(x_1) < 0$

Polynomials (like f) are continuous,

so by IVT, since $f(x_1) < 0$ and $f(x_2) > 0$,

there is a point r, $x_1 < r < x_2$ such that

$f(r) = 0$. $\therefore$ f has precisely one root.