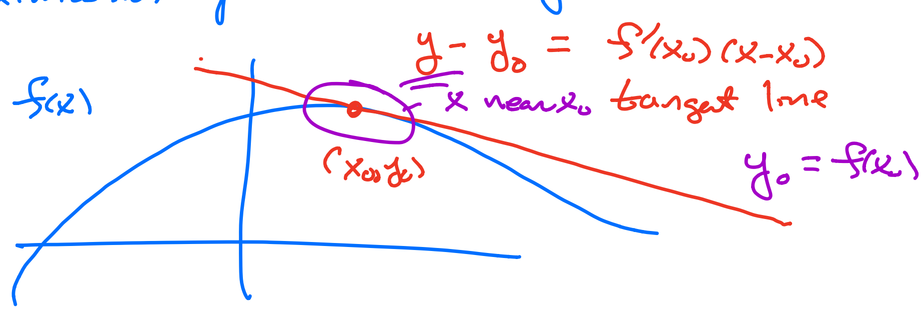
Recitation Calculus I (Adamski)
Tuesday, December 1, 2020

- e^x, $\ln(x)$, b^x, $\log_b(x)$ integrals.

Approximation of a curve by its tangent line

$y = f(x)$



y-coord of curve: $\underline{f(x)}$

y-coord of tangent line: $\underline{f(x_0) + f'(x_0)(x - x_0)}$

Fundamental Approximation.

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0) \quad \text{for } x \text{ near } x_0$$

Ex $f(x) = \sqrt{1+x}$ $\quad (x_0, y_0) = (0, 1)$

$y_0 = f(x_0) = f(0) = \sqrt{1} = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$; $m = f'(x_0) = f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$

$\therefore \sqrt{1+x} \simeq 1 + \frac{1}{2}(x - 0)$ for x near 0

$\sqrt{1+x} \simeq 1 + \frac{1}{2}x$ for small x $\qquad x = 0.01$

$\sqrt{1+.01} \simeq 1 + \frac{1}{2}(.01) = 1.005$

$1.004987\ldots$ ✓

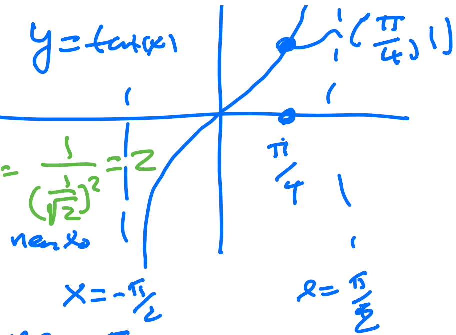
$\sqrt{1.01} \simeq$

Ex $f(x) = \tan(x)$

$f'(x) = \sec^2(x)$

$(x_0, y_0) = (\tfrac{\pi}{4}, 1)$

$f'(x_0) = \sec^2(x_0) = \sec^2(\pi/4) = \dfrac{1}{\cos^2(\pi/4)} = \dfrac{1}{(\frac{1}{\sqrt{2}})^2} = 2$



$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$ for x near x_0

$\tan(x) \simeq 1 + 2(x - \pi/4)$ for x near $\pi/4 \simeq 0.785398\ldots$

Choose $x = 3/4 = 0.75$ near $\pi/4 \simeq 0.78$

$\tan(3/4) \simeq 1 + 2(\tfrac{3}{4} - \tfrac{\pi}{4}) \simeq 0.9292036$

$\tan(3/4) \simeq 0.9315\ldots$

2 (dec)

Ex $f(x) = \ln(a + \frac{x}{a})$ x variable

$f(0) = \ln(a + 0) = \ln(a)$ $\underline{a} > 0$ positive parameter.

$(x_0, y_0) = (0, \ln(a))$

$f'(x) = \dfrac{1}{a + \frac{x}{a}} \left(\dfrac{1}{a}\right) = \dfrac{1}{a^2 + x}$

$\dfrac{d}{dx}\left(a + \dfrac{x}{a}\right) = 0 + \dfrac{1}{a}$; $f'(x_0) = f'(0) = \dfrac{1}{a^2}$

$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$ for x near x_0

$$\boxed{\ln\left(a + \frac{x}{a}\right) \simeq \ln(a) + \frac{1}{a^2}(x - 0) \text{ for } x \text{ near } 0.}$$

$\dfrac{d}{dx} e^x = e^x$; $\dfrac{d}{dx} \ln(x) = \dfrac{1}{x}$ ☆

These two go together, at least if you use the chain rule.

$e^{\ln(x)} = x$ for $x > 0$; $\ln(e^x) = x$ for $x \in \mathbb{R}$

Suppose $g(\underline{\ln(x)}) = x$; $g'(\ln(x))\left(\frac{1}{x}\right) = 1$

$\therefore$ $g'(\ln(x)) = x = g(\ln(x))$ Let $y = \ln(x)$

$\therefore$ $g'(y) = g(y)$

$\therefore$ Any g is its own derivative if it is the inverse function of $\ln(x)$.

Suppose $f(e^x) = x$; $f'(e^x)\,e^x = 1$

$\therefore$ $f'(e^x) = \dfrac{1}{e^x}$ Let $y = e^x$

$\therefore$ $f'(y) = \dfrac{1}{y}$.

Any $f(y)$ has derivative $\frac{1}{y}$ if f is the inverse function of e^x.

$\frac{d}{dx}\ln(3x) = \frac{1}{x} = \frac{d}{dx}\ln(x)$

$C + \ln(x) = \underline{\ln(3) + \ln(x) = \ln(3x)}$

OK - derivatives should be equal.

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln(x) = \underline{\underline{\frac{1}{x}}}$$

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}$$

$$☆\ \frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$\frac{d}{dx}\ln(3x) = \frac{1}{3x}3 = \underline{\underline{\frac{1}{x}}}$$

$$\frac{d}{dx}e^{3x} = e^{3x}(3) = 3e^{3x}$$

$$\frac{d}{dx}\ln(\tan(x)) = \frac{\sec^2(x)}{\tan(x)}$$

$$\frac{d}{dx}e^{\tan(x)} = e^{\tan(x)}\sec^2(x)$$

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x} \quad \Bigg| \quad \frac{d}{dx}\ln(|x|) = \frac{1}{x}}$$

$\underline{Ex}$ $f(x) = \ln(|\tan(x) + \sec(x)|)$

$$f'(x) = \frac{\sec^2(x) + \tan(x)\sec(x)}{\tan(x) + \sec(x)} = \sec(x)\frac{(\sec(x) + \tan(x))}{\tan(x) + \sec(x)}$$

$= \sec(x)$

Remember that

$\therefore \int \sec(x)\,dx = \ln(|\tan(x) + \sec(x)|) + C.$

Ex $\frac{1}{3}\int 3e^{3x}\,dx =$

Let $u = 3x$
$du = 3\,dx$
$\frac{1}{3}du = dx$

$\int e^{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int e^{u}\,du = \frac{1}{3}e^{u} + C$ Check.

$\therefore \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

$\frac{d}{dx}\frac{1}{3}e^{3x} = e^{3x}\frac{3}{3}$
$= e^{3x}$

Ex $\int e^{ax}\,dx$ $a \geq 0$

$= \frac{1}{a}e^{ax} + C$

$\frac{d}{dx}\left(\frac{1}{a}e^{ax} + C\right) = \frac{1}{a}e^{ax}(a) + 0$
$= e^{ax}$ ✓.

Ex $\int x e^{x^2}\,dx$

Let $u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$= \int e^{u}\frac{1}{2}du = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + C.$

$= \frac{1}{2}e^{x^2} + C.$

Ex $\int \frac{e^{x}}{1+e^{x}}\,dx$

Let $u = 1 + e^{x}$
$du = e^{x}\,dx$

$= \int \frac{du}{u} = \ln(|u|) + C = \ln(|1+e^{x}|) + C$
$= \ln(1+e^{x}) + C.$ $1 = \frac{x^1}{x^1} = x^{1-1} = x^0$

Check

$\frac{d}{dx}\left(\ln(1+e^{x})\right) = \frac{0+e^{x}}{1+e^{x}} = \frac{e^{x}}{1+e^{x}}$ ✓.

$e^{x}e^{-x} = e^{x-x}$

$= e^0 = 1$

Ex $\displaystyle\int \frac{1}{1+e^{-x}}\,dx = \int \left(\frac{e^x}{e^x}\right)\frac{1}{1+e^{-x}}\,dx$

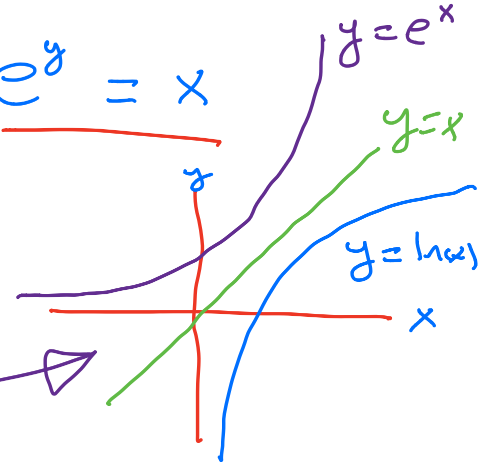
$\displaystyle = \int \frac{e^x}{e^x+1}\,dx = \ln(1+e^x) + C.$

as before.

$y = \ln(x) \iff \underline{e^y = x}$

$e^y = e^{\ln(x)} = x$

Geometric Interpretation of inverse functions



$\displaystyle\int \frac{u'}{u}\,dx = \ln(|u|) + C$

$\displaystyle\int \frac{x}{(1+x^2)^3}\,dx$

Let $\underbrace{u = 1+x^2}$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$