Recitation Calculus One (Adamski)

Tuesday, November 10, 2020

- Fundamental Th of Calculus §4.3

FTC

① Assume $f: [a,b] \to \mathbb{R}$ is a differentiable function and f' is a continuous function on $[a,b]$. Then

$$(\star)\quad \underbrace{\int_a^b f'(x)\,dx}_{\text{LHS}} = \underbrace{f(x)}_{\text{RHS}}\Big|_a^b = f(b) - f(a)$$

② Assume $f: [a,b] \to \mathbb{R}$ is a continuous function.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Need $f'(x) = x^2$
$a = 1;\ b = 3;$
$f(x) = \frac{1}{3}x^3 + C$

Ex① $\displaystyle\int_1^3 x^2\,dx = \left(\frac{1}{3}x^3 + C\right)\Big|_1^3$

$= \left(\frac{1}{3}3^3 + C\right) - \left(\frac{1}{3}1^3 + C\right) = \frac{3^3-1}{3} = \frac{26}{3}$

Ex② $\displaystyle\frac{d}{dx}\int_0^x t^2\,dt = \underbrace{\frac{d}{dx}\left\{\frac{t^3}{3}\Big|_0^x\right\} = \frac{d}{dx}\left(\frac{x^3}{3} - \frac{0^3}{3}\right) = x^2}_{\text{not needed}}$

$= x^2$ by FTC

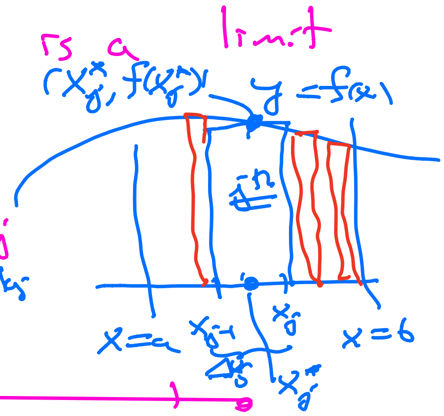
$$\frac{d}{dx}\int_0^x \frac{\sin(t)}{1+t^2}\,dt = \frac{\sin(x)}{1+x^2} \text{ by FTC}$$

$$\int t^2\,dt = \frac{t^3}{3} + C$$ (Need constant) indefinite integral

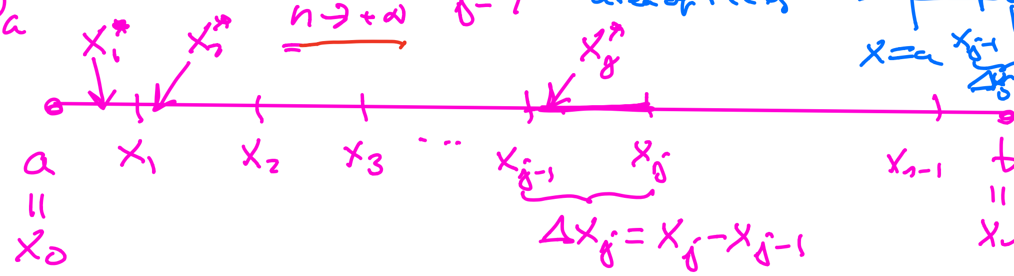
$$\int_1^3 t^2\,dt = \left\{ \frac{t^3}{3} \right|_1^3 = \frac{3^3 - 1^3}{3}$$ (Do not need C) definite integral

(has limits a, b)

A definite integral $\int_a^b f(x)\,dx$ is a limit of Riemann sums.



$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{j=1}^{n} \underbrace{f(x_j^*)\Delta x_j}_{\text{area of rects}}$$



(Practical Answer)

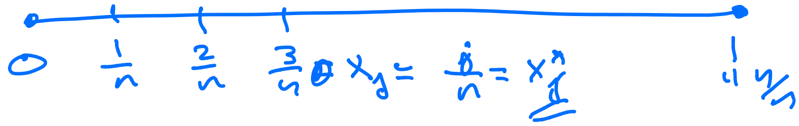
Ex What is the value (approximately) of

$$\int_0^1 \sqrt{1 + x^3}\,dx \; ? = \sum_{j=1}^{n} \sqrt{1 + \left(\tfrac{j}{n}\right)^3}\left(\tfrac{1}{n}\right) \quad n = 100.$$

$$= \sqrt{1 + \left(\tfrac{0}{100}\right)^3}\,\tfrac{1}{100} + \sqrt{1 + \left(\tfrac{1}{100}\right)^3}\,\tfrac{1}{100} + \cdots + \sqrt{1 + \left(\tfrac{100}{100}\right)^3}\,\tfrac{1}{100}$$

Do you know a function $f(x)$ such that

$$f'(x) = \sqrt{1 + x^3} \;?$$



Ex $\frac{d}{dx}\left(\int_1^x \frac{\sin(t)}{1+t^4}\,dt\right) = \frac{\sin(x)}{1+x^4}$ by FTC.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

$$\frac{d}{dx}\left(\int_x^a f(t)\,dt\right) = \frac{d}{dx}\left(-\int_a^x f(t)\,dt\right) =$$

$$= -\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = -f(x)$$

$$\frac{d}{dx}\left(\int_a^{x^2} f(t)\,dt\right) = \boxed{f(x^2)\,(2x)}$$

$t = x^2$ (2x)

This requires some explanation (Chain Rule)

$G(x) = \int_a^x f(t)\,dt$; $\boxed{G'(x) = f(x).}$

$G(10) = \int_a^{10} f(t)\,dt$

$G(y) = \int_a^y f(t)\,dt$

$G(x^2) = \boxed{\int_a^{x^2} f(t)\,dt}$

$$\int_a^{x^2} f(t)\,dt = G(x^2)$$

$$\frac{d}{dx}\int_a^{x^2} f(t)\,dt = \frac{d}{dx}G(x^2)$$

$$= G'(x^2)(2x) \quad \text{by Chain Rule}$$

$$= f(x^2)\,(2x)$$

$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

$(x^3)^2 = x^6$

Ex $\frac{d}{dx}\int_0^{\sin(x)} \sqrt{1+t^3}\,dt =$

$\left(\sqrt{1+\sin^3(x)}\right)\cos(x)\,.\ (1)$

Ex $\frac{d}{dx}\int_0^{x^3} \frac{\cos(t)}{1+\sin(t^2)}\,dt = \frac{\cos(x^3)}{1+\sin(x^6)}\,3x^2$

Ex $\frac{d}{dx}\int_{x}^{x^2} \sin(t^2)\,dt =$ $\quad \frac{d}{dx}x^2 = 2x \quad t := x^2 \quad \frac{d}{dx}x = 1$

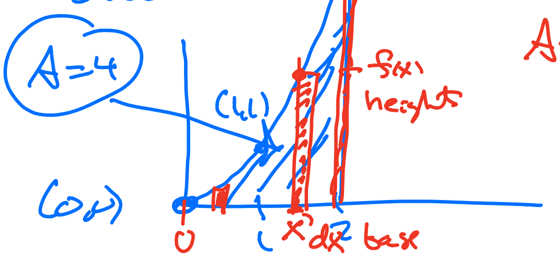
$= \sin(x^4)(2x) - \sin(x^2)(1)$

$= 2x\sin(x^4) - \sin(x^2)$

FTC $\int_a^b \underline{f'(t)}\,dt = f(t)\Big|_a^b = f(b) - f(a)$

(2,8)

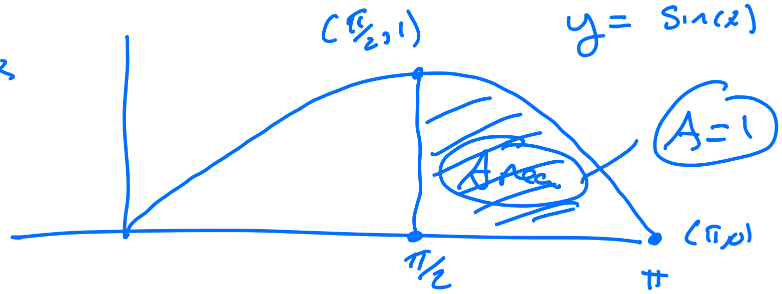
Find the area under the curve $y = x^3$
over the interval [0,2]



Area $= \int_0^2 \underline{f(x)}\,\underline{dx} = \int_0^2 \underline{x}^3\,dx$

$= \left\{\frac{x^4}{4}\right|_0^2 = \frac{2^4 - 0^4}{4} = 4$

Ex



Find area under $y = \sin(x)$ over the interval $[\frac{\pi}{2}, \pi]$.

$A = \int_{\pi/2}^{\pi} \underline{\sin(x)}\,\underline{dx} = \left\{-\cos(x)\right|_{\pi/2}^{\pi} = -\cos(\pi) - -\cos(\frac{\pi}{2})$

$= -(-1) + 0 = 1$

to, from, of, don't forget