$$\int x^3 \underbrace{\sqrt{x^2+a}}_{\sqrt{u}}\, dx$$

Let $u = x^2 + a$ → $x^2 = u - a$

EXPRESSION INSIDE ANOTHER FUNCTION

$du = 2x\, dx$

$\frac{1}{2}\, du = x\, dx$

$$\int \underbrace{x^2}_{u-a} \underbrace{\sqrt{x^2+a}}_{\sqrt{u}} \underbrace{x\, dx}_{\frac{1}{2}du}$$

$$\rightsquigarrow \frac{1}{2}\int (u-a)u^{1/2}\, du$$

ALL u's NO x's

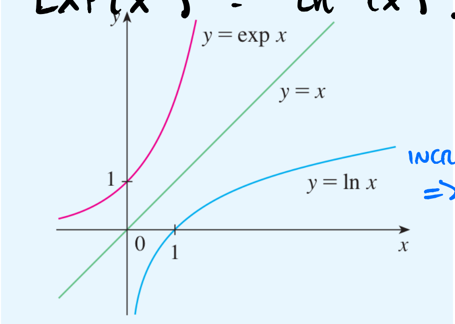
$$= \frac{1}{2}\int u^{3/2} - au^{1/2}\, du$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}au^{3/2}\right] + C$$

$$\rightsquigarrow \boxed{\frac{1}{5}(x^2+a)^{5/2} - \frac{1}{3}a(x^2+a)^{3/2} + C}$$

FINAL EXAM : MONDAY 12/14 1:30 - 3:30 PM (PAPER & ZOOM)

**Def:** $\text{Exp}(x) = \text{Ln}^{-1}(x)$.




INCREASING $\Rightarrow$ 1:1

$$\begin{tikzcd} a \arrow[r, bend left, "\text{Ln}(a)=b"] & b \arrow[l, bend left, "a=\text{Exp}(b)"] \end{tikzcd}$$

$\text{DOM}(\text{Exp}) = \text{RAN}(\text{Ln}) = (-\infty, \infty)$

$\text{RAN}(\text{Exp}) = \text{DOM}(\text{Ln}) = (0, \infty)$

CANCELLATION LAWS:

$\text{Exp}(\ln x) = x \,, \quad x > 0$

$\text{Ln}(\text{Exp}(x)) = x$

IMPORTANT VALUES:

$\text{Exp}(0) = 1 \qquad (\text{Ln}(1) = 0)$

$\text{Exp}(1) = e \qquad (\text{Ln}(e) = 1)$

**THM:** $\text{Exp}(x) = e^x$.

**PROOF:** $\text{Ln}(e^x) = x\,\underbrace{\text{Ln}(e)}^{1} = x$ BY Log RULE 3.

CANCELLATION LAW

$\therefore \quad \cancel{\text{Exp}}(\cancel{\text{Ln}}(e^x)) = \text{Exp}(x)$

$e^x = \text{Exp}(x)$

$\square$

$$\begin{tikzcd} a \arrow[r, bend left, "\text{Ln}(a)=b"] & b \arrow[l, bend left, "a=e^b"] \end{tikzcd}$$

**ex.** SOLVE THE INEQUALITY FOR $x$:

$$5 < e^{4x-3} \leq 7$$

$\text{Ln}(x)$ IS INCREASING FUNCTION:

$a < b \Rightarrow \text{Ln}(a) < \text{Ln}(b)$

Ln PRESERVES ORDER

$$\text{Ln}(5) < \cancel{\text{Ln}}(\cancel{e}^{4x-3}) \leq \text{Ln}(7)$$

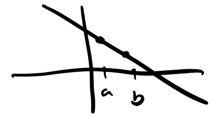
$$\text{Ln}(5) < 4x-3 \leq \text{Ln}(7)$$

$$\boxed{\frac{\text{Ln}(5)+3}{4} < x \leq \frac{\text{Ln}(7)+3}{4}}$$

NOTE: BEFORE TAKING Ln OF BOTH SIDES OF AN EQUATION, YOU NEED TO BE SURE THAT BOTH SIDES ARE POSITIVE.

$f(x) = -x$

$a < b$




$-a > -b$

Dom$(f)$: $10 - e^{3x} \geq 0$ SOLVE FOR $x$

$10 \geq e^{3x}$

$\ln(10) \geq \ln(e^{3x}) = 3x$

$\frac{1}{3}\ln(10) \geq x$

Dom$(f) = (-\infty, \frac{1}{3}\ln 10]$

**ex.** FIND THE DOMAIN OF $f$ AND $f^{-1}$.

$$f(x) = \sqrt{10 - e^{3x}}$$

FIND $f^{-1}$: $y = \sqrt{10 - e^{3x}}$ SOLVE FOR $x$.

$$y^2 = 10 - e^{3x}$$

$$e^{3x} = 10 - y^2$$

TO BRING VARIABLES OUT OF EXPONENT POSITION, TAKE ln OF BOTH SIDES.

$$\cancel{\ln}(\cancel{e}^{3x}) = \ln(10 - y^2)$$

$$3x = \ln(10 - y^2) \longrightarrow x = \frac{1}{3}\ln(10 - y^2)$$

$$y = \frac{1}{3}\ln(10 - x^2)$$

SWITCH: $x \leftrightarrow y$

$$\boxed{f^{-1}(x) = \frac{1}{3}\ln(10 - x^2)}$$

DOMAIN OF $f^{-1}$: $10 - x^2 > 0 \implies 10 > x^2 \implies |x| < \sqrt{10}$

DOMAIN OF $f^{-1}$: $(-\sqrt{10}, \sqrt{10})$

**ex.** FIND $f^{-1}(x)$ IF $f(x) = \dfrac{1 - e^{-x}}{1 + e^{-x}}$.

$y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$ SOLVE FOR $x$.

$$y(1 + e^{-x}) = 1 - e^{-x}$$

$$y + ye^{-x} = 1 - e^{-x}$$

$$e^{-x} + ye^{-x} = 1 - y$$

$$e^{-x}(1 + y) = 1 - y$$

$$e^{-x} = \frac{1-y}{1+y}$$

$$\ln e^{-x} = \ln \frac{1-y}{1+y}$$

$$-x = \ln \frac{1-y}{1+y}$$

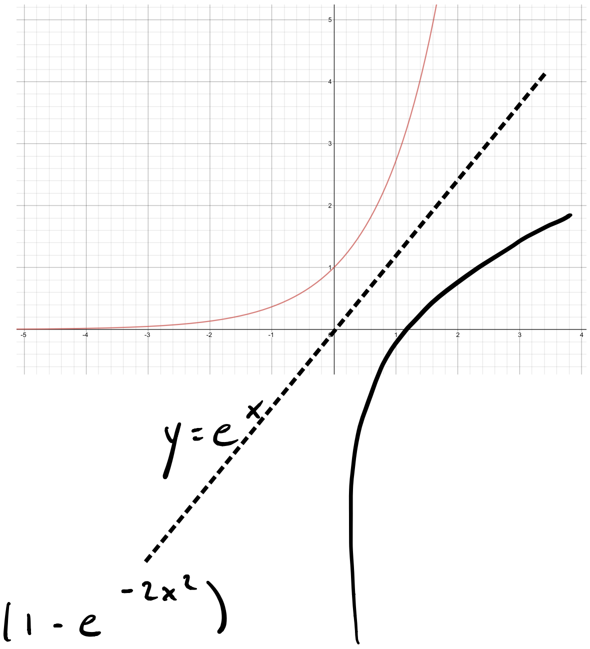
$$x = -\ln \frac{1-y}{1+y}$$

$$\boxed{f^{-1}(x) = -\ln \frac{1-x}{1+x}}$$

**6 Properties of the Natural Exponential Function** The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all $x$. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the $x$-axis is a horizontal asymptote of $f(x) = e^x$.




**ex.** FIND THE LIMIT $\displaystyle\lim_{x \to \infty} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$

$$\lim_{x \to \infty} \frac{\cancel{e^{x^2}}\left(1 - e^{-2x^2}\right)}{\cancel{e^{x^2}}\left(1 + e^{-2x^2}\right)} = \frac{\displaystyle\lim_{x \to \infty}\left(1 - e^{-2x^2}\right)}{\displaystyle\lim_{x \to \infty}\left(1 + e^{-2x^2}\right)}$$

$$= \frac{1 - 0}{1 + 0} = \boxed{1}$$

$$\lim_{x \to \infty} \frac{e^{x^2} - e^{-2x^2}}{e^{x^2} + e^{-3x^2}}$$

**ex.** FIND THE LIMIT $\displaystyle\lim_{x \to 0} e^{-\frac{1}{x^2}} \sin\left(\frac{1}{x^2}\right)$

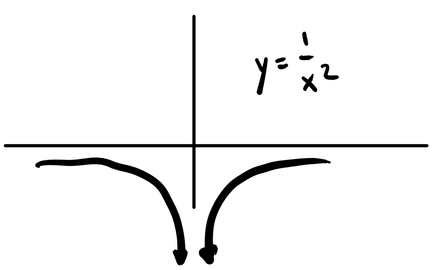
$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$$

$$-e^{-1/x^2} \le e^{-\frac{1}{x^2}} \sin\left(\frac{1}{x^2}\right) \le e^{-1/x^2}$$

$$\lim_{x \to 0} e^{-1/x^2} = \lim_{u \to -\infty} e^{u} = 0$$

$\downarrow 0 \qquad \downarrow 0 \qquad \downarrow 0$

BY SQUEEZE THEOREM.




LET $u = -1/x^2$

AS $x \to 0$, $u \to -\infty$

**7** **Laws of Exponents** If $x$ and $y$ are real numbers and $r$ is rational, then

**1.** $e^{x+y} = e^x e^y$ **2.** $e^{x-y} = \dfrac{e^x}{e^y}$ **3.** $(e^x)^r = e^{rx}$

PROOF OF 3.

$\ln((e^x)^r) = r \ln(e^x)$ LOG RULE 3

$= rx$ DEFINITION

$e^{\ln((e^x)^r)} = e^{rx}$

$(e^x)^r = e^{rx}$ CANCELLATION RULE.

$\square$

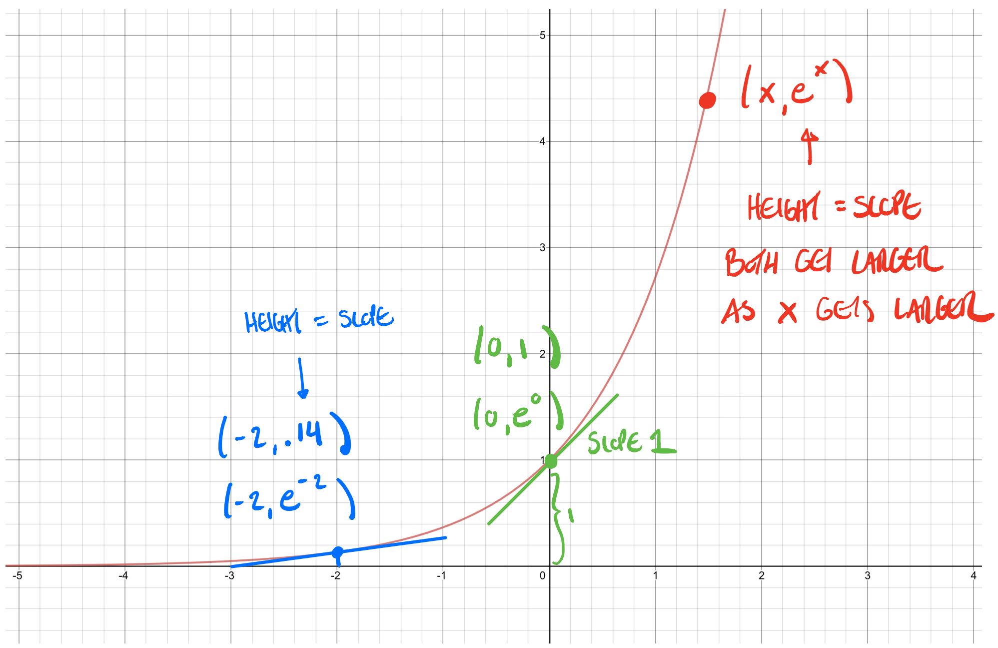
$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

PROOF: LET $y = e^x$. FIND $y'$.

$\ln y = x$. $y$ IS DEFINED IMPLICITLY

IMPLICIT DIFFERENTIATION

$$\frac{1}{y}\, y' = 1 \quad \Rightarrow \quad y' = y = e^x \qquad \square$$




AT ANY POINT ON THE GRAPH

$y = e^x$

THE SLOPE = THE HEIGHT

$y' = y$

SPEEDOMETER ODOMETER

THING ABOUT $s'(t) = s(t)$ POSITION AT TIME $t$ ALONG STRAIGHT PATH

i.e. $v(t) = s(t)$

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

ex. DIFFERENTIATE THE FUNCTION $f(x) = x^2 e^{-1/x}$

$$f'(x) = 2x \cdot e^{-1/x} + x^2 \cdot e^{-1/x} \cdot \frac{d}{dx}\left[-x^{-1}\right]$$

$$f'(x) = 2x \cdot e^{-1/x} + \cancel{x^2} \cdot e^{-1/x} \cdot \cancel{x^{-2}}$$

$$\boxed{f'(x) = (2x+1)e^{-1/x}}$$

**56.** Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

Hvhv

$$\frac{d}{dx}\left[xe^y + ye^x\right] = \frac{d}{dx}\left[1\right]$$

$$1e^y + xe^y\frac{dy}{dx} + \frac{dy}{dx}e^x + ye^x = 0$$

$$\frac{dy}{dx}\left(xe^y + e^x\right) = -e^y - ye^x$$

$$\boxed{y - 1 = (-e-1)x}$$

$$\frac{dy}{dx} = \frac{-e^y - ye^x}{xe^y + e^x} \xrightarrow[\text{AT } (0,1)]{} -e-1$$

SLOPE

THRU $(0,1)$

**59.** For what values of $r$ does the function $y = e^{rx}$ satisfy the differential equation $\underbrace{y'' + 6y' + 8y = 0}_{\text{DIFFERENTIAL EQUATION}}$?

$r$ CONSTANT

$y = e^{rx}$

$y' = e^{rx} \cdot \frac{d}{dx}[rx] = e^{rx} \cdot r = re^{rx}$

$y'' = r \cdot \frac{d}{dx}[e^{rx}] = r \cdot re^{rx} = r^2 e^{rx}$

$y'' + 6y' + 8y = 0 \longrightarrow r^2 e^{rx} + 6re^{rx} + 8e^{rx} = 0$

$e^{rx}(r^2 + 6r + 8) = 0$

$e^{rx}(r+4)(r+2) = 0$

$e^{rx} = 0$ ⋮ NO SOLUTIONS

$r + 4 = 0$ $\qquad$ $r + 2$

$\boxed{r = -4 \qquad r = -2}$

$e^x > 0$

$\text{RAN}(e^x) = (0, \infty)$

TWO SOLUTIONS: TO DIFF. EQ.

$y = e^{-2x}$

$y = e^{-4x}$

$$\int e^x \, dx = e^x + C$$

EX. EVALUATE THE INTEGRAL

(a) $\displaystyle\int \frac{e^x}{(1-e^x)^2} \, dx$

(b) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2} \, dx$

(c) $\displaystyle\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}$