FINAL EXAM MON 12/14 1:30 - 3:30 PM

FINAL LECTURE MON 12/7

FINAL RECITATION TUE 12/8

Find the derivative of the function.

$$g(x) = \int_{\tan x}^{4x^2} \frac{1}{\sqrt{5 + t^4}}\, dt$$

$g'(x) =$

LET $F(x)$ BE ANTI-DERIV. OF $\frac{1}{\sqrt{5+t^4}}$.

THEN $F'(x) = \frac{1}{\sqrt{5+t^4}}$

ALSO, $g(x) = F(x)\Big|_{\tan x}^{4x^2}$

i.e. $g(x) = F(4x^2) - F(\tan x)$

$$g'(x) = F'(4x^2)(8x) - F'(\tan x)(\sec^2 x)$$

$$\boxed{g'(x) = \frac{1}{\sqrt{5 + (4x^2)^4}}\, 8x - \frac{1}{\sqrt{5 + (\tan x)^4}}\, \sec^2 x}$$

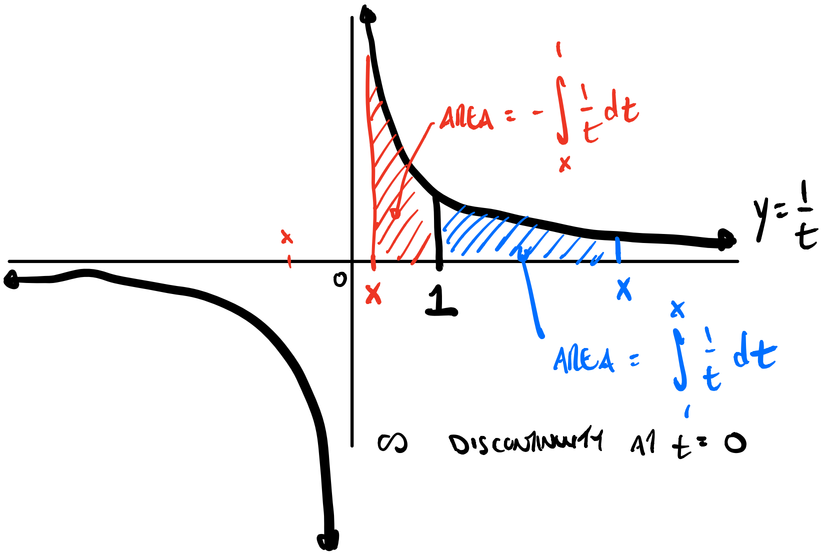
# §6.2* The Natural Logarithm

**1 Definition** The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

- Domain
- Pos, Neg, 0
- Increasing (F.T.C.)
- Concave Down
- $\lim_{x \to \infty} \ln x = \infty$

  $\lim_{x \to 0^+} \ln x = -\infty$




$\text{Domain}(\ln) = (0, \infty)$

$\ln x > 0$ IF $x > 1$

$\ln x = 0$ IF $x = 1$ $\left( \int_1^1 \frac{1}{t}\,dt = 0 \right)$

$\ln x < 0$ IF $x < 1$

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t}\,dt = \frac{1}{x} \qquad \text{(FTC.)}$$

SINCE $\frac{1}{x} > 0$ WHEN $x > 0$,

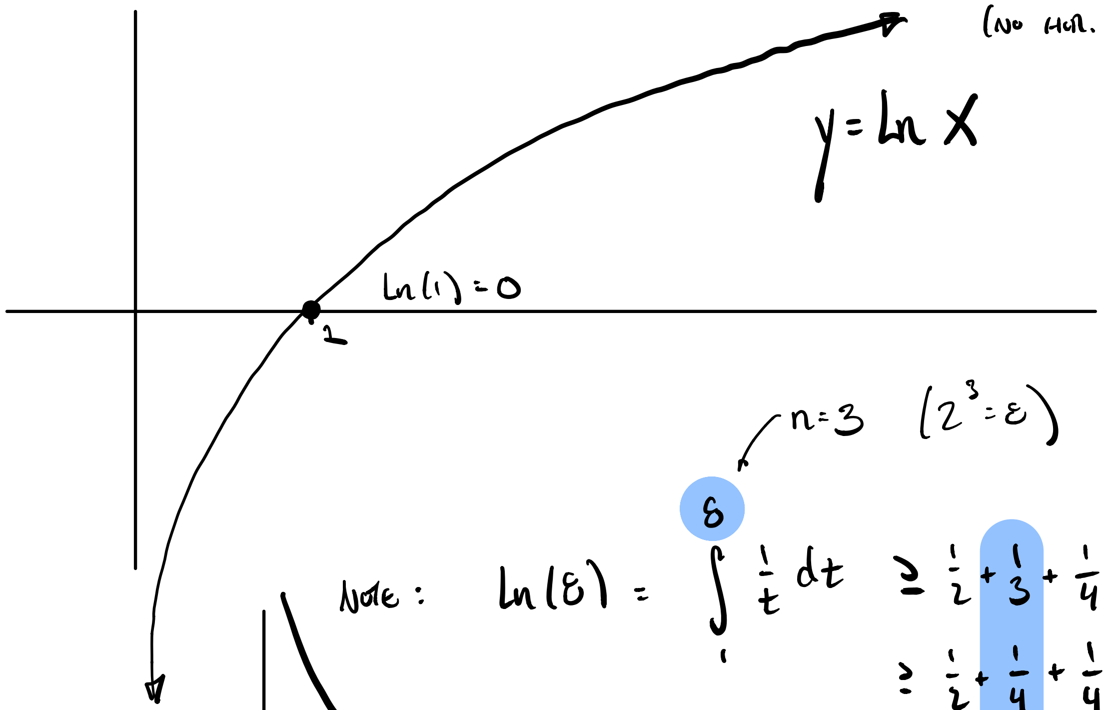
$\left( \frac{d}{dx} \ln x > 0 \right) \Rightarrow$ **$\ln x$ is increasing on Domain.**

$$\frac{d^2}{dx^2}\left[\ln x\right] = \frac{d}{dx}\left[\frac{1}{x}\right] = -x^{-2} = \frac{-1}{x^2} < 0$$

$\therefore$ GRAPH OF $y = \ln x$ IS CONCAVE DOWN ON DOMAIN

(NO HOR. ASYMPTOTE)

SLOW GROWTH

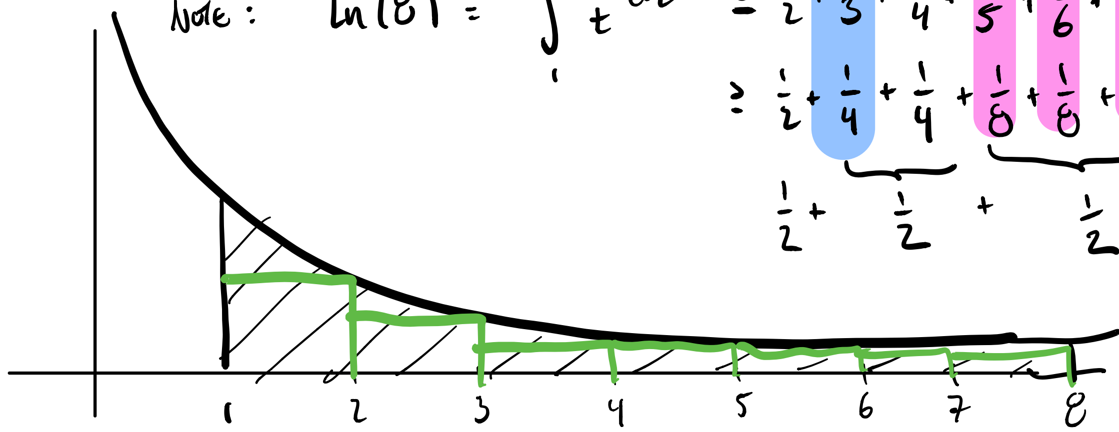



$n = 3$ $(2^3 = 8)$

NOTE: $\ln(8) = \int_1^8 \frac{1}{t}\,dt \ge \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$$\ge \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}}$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$




$$\Rightarrow \ln(2^n) = \int_1^{2^n} \frac{1}{t}\,dt \ge \frac{n}{2}$$

$$\lim_{x\to\infty} \ln(x) = \lim_{n\to\infty} \ln(2^n) \ge \lim_{n\to\infty} \frac{n}{2} = \infty$$

$$\boxed{\lim_{x\to\infty} \ln(x) = \infty, \quad \lim_{x\to 0^+} \ln(x) = -\infty}$$

(SIMILAR ARGUMENT)

SINCE $\frac{d}{dx}\ln x = \frac{1}{x} > 0$ FOR $x > 0$, $\ln x$ INCREASING

$\Rightarrow$ $\ln x$ IS ONE TO ONE

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad \text{IF } n \neq -1$$

$$\int x^{-1}\,dx = \ln x$$

MISSING PIECE

**81.** If $g$ is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

INVERSE FUNCTION THM: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$$g'(2) = (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

WE NEED $f'(x) = 2 + \frac{1}{x}$ & $f^{-1}(2) = 1$

$f(1) = 2$

$f(1) = 2(1) + \ln 1 = 2 + 0 = 2$ ✓

$$g'(2) = (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{2 + \frac{1}{(1)}} = \frac{1}{3}$$

**3 Laws of Logarithms** If $x$ and $y$ are positive numbers and $r$ is a rational number, then

**1.** $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

PROOF OF 1: LET $f(x) = \ln(ax)$, $a$ IS POSITIVE CONSTANT.

$$f'(x) = \frac{1}{ax} \cdot a = \frac{1}{x} = \frac{d}{dx}[\ln x]$$

$f(x)$ & $\ln(x)$ HAVE SAME DERIVATIVE.

$f(x) = \ln(x) + C$ FOR SOME CONSTANT $C$.

$\downarrow$

$\ln(ax) = \ln(x) + C$ PLUG IN $x = 1$: $\ln(a) = \ln 1 + C$

$\ln(a) = 0 + C$

$\therefore \ln(ax) = \ln(a) + \ln(x)$ TRUE FOR ANY $a > 0$. $\square$

ex. EXPAND THE LOGARITHMIC EXPRESSION $\ln\left(\frac{4a^2b^3}{\sqrt{a^2+b^2}}\right)$

$$\ln\left(\frac{4a^2b^3}{\sqrt{a^2+b^2}}\right) = \ln(4a^2b^3) - \ln\left((a^2+b^2)^{1/2}\right) \quad (2)$$

**3 Laws of Logarithms** If $x$ and $y$ are positive numbers and $r$ is a rational number, then

**1.** $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

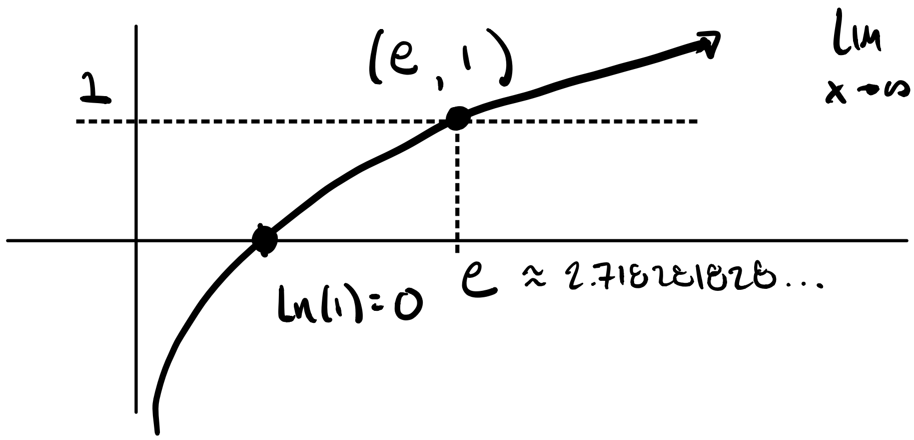
PRODUCTS, QUOTIENTS, EXPONENTS

$$= \ln 4 + \ln(a^2) + \ln(b^3) - \ln\left((a^2+b^2)^{1/2}\right) \quad (1)$$

$$\boxed{= \ln 4 + 2\ln(a) + 3\ln(b) - \frac{1}{2}\ln(a^2+b^2)}$$

sum

**5 Definition** $e$ is the number such that $\ln e = 1$.




$\lim\limits_{x \to \infty} \ln x = \infty$

INTERMEDIATE VALUE THM:

$\exists \, \#$ s.t. $\ln \# = 1$

WE CALL THIS # $e$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

ex. $\dfrac{d}{dx} \ln(3x^4 - 4x^3)$ (TWO WAYS)

$$= \frac{12x^3 - 12x^2}{3x^4 - 4x^3} \checkmark$$

$$\frac{d}{dx}\left[\ln\left(x^3(3x-4)\right)\right]$$

$$= \frac{d}{dx}\left[\ln(x^3) + \ln(3x-4)\right] \quad (1)$$

$$= \frac{d}{dx}\left[3\ln(x) + \ln(3x-4)\right]$$

$$\frac{3}{x} + \frac{3}{3x-4}$$

SAME!

ex. $\dfrac{d}{dx} \ln\left(\dfrac{x^2-1}{\sqrt{x^2+1}}\right)$ (PRECALC BEFORE CALC)

$$\frac{1}{\frac{x^2-1}{\sqrt{x^2+1}}} \cdot \frac{d}{dx} \frac{x^2-1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left[ \ln(x^2-1) - \frac{1}{2} \ln(x^2+1) \right]$$

$$\frac{2x}{x^2-1} - \frac{\cancel{2}x}{\cancel{2}(x^2+1)}$$

DOMAIN $(-\infty,0)\cup(0,\infty)$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

CHECK:

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{d}{dx}\ln x = \frac{1}{x} & \text{IF } x>0 \\ \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x} & \text{IF } x<0 \end{cases}$$

ANTI-DERIV. OF $\frac{1}{x}$

**ex.** $\frac{\ln x}{x}\,dx$

LET $u = \frac{\ln x}{x}$ $\qquad du = \frac{x\frac{1}{x} - \ln}{x^2}\,dx$

LET $u = \ln x$ $\qquad du = \frac{1}{x}\,dx$

$\ln x \; \frac{1}{x}\,dx$

$$\int_{x=1}^{x=e} \xrightarrow{u=\frac{\ln x}{x}} \int_{u=\frac{\ln 1}{1}=0}^{u=\frac{\ln e}{e}=\frac{1}{e}} u\,du = \frac{1}{2}u^2\Big|_0^{1/e}$$

$$= \frac{1}{2}\left(\left(\frac{1}{e}\right)^2 - (0)^2\right)$$

$$= \frac{1}{2e^2}$$

ex. $\int \tan x \, dx$

**65–74** Evaluate the integral.

**65.** $\int_2^4 \frac{3}{x}\,dx$

**66.** $\int_0^3 \frac{dx}{5x+1}$

**67.** $\int_1^2 \frac{dt}{8-3t}$

**68.** $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

**69.** $\int_1^e \frac{x^2+x+1}{x}\,dx$

**70.** $\int_e^6 \frac{dx}{x \ln x}$

**71.** $\int \frac{(\ln x)^2}{x}\,dx$

**72.** $\int \frac{\cos x}{2+\sin x}\,dx$

**73.** $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

**74.** $\int \frac{\cos(\ln t)}{t}\,dt$

### Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to $x$.
3. Solve the resulting equation for $y'$.

**61–64** Use logarithmic differentiation to find the derivative of the function.

**61.** $y = (x^2 + 2)^2(x^4 + 4)^4$

**62.** $y = \dfrac{(x + 1)^4(x - 5)^3}{(x - 3)^8}$

**63.** $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

**64.** $y = \dfrac{(x^3 + 1)^4 \sin^2 x}{x^{1/3}}$