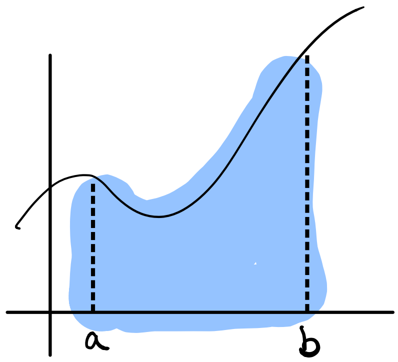
4.4 The Indefinite Integral & The Net Change Thm.

Def: The indefinite integral $\int f(x)\,dx$

is used to represent the entire collection of antiderivatives of $f(x)$. Thus, if $F'(x) = f(x)$ then

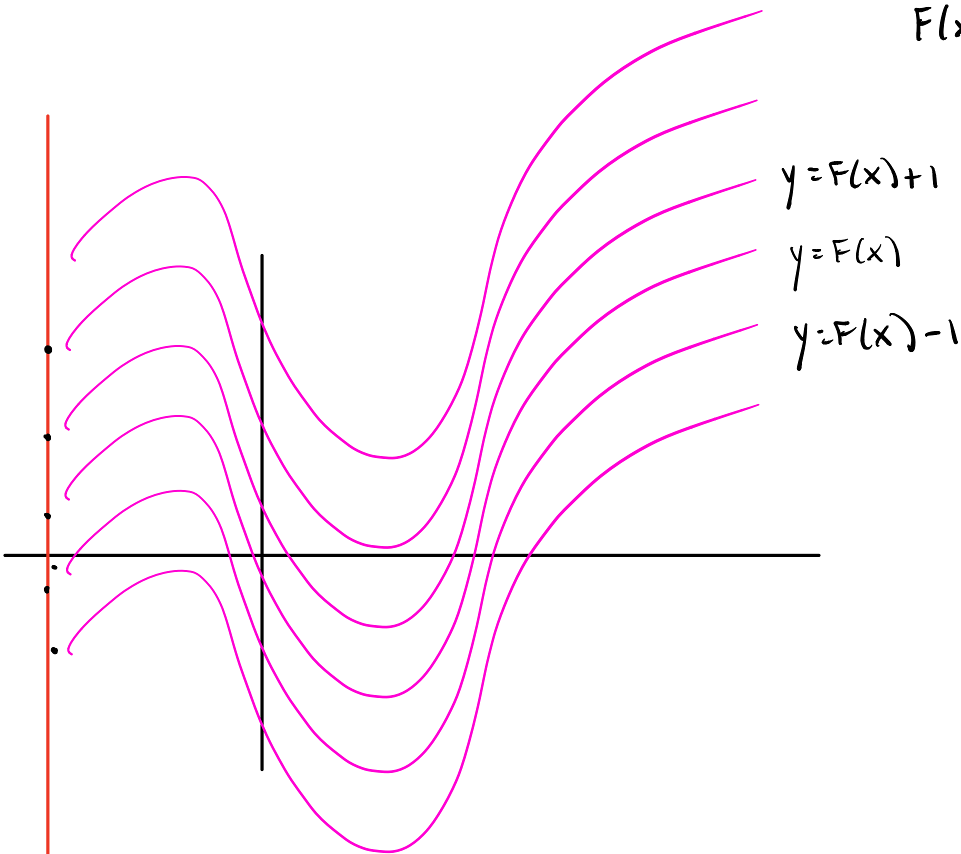
$$\int f(x)\,dx = F(x) + C.$$

Note: Definite integral $\int_a^b f(x)\,dx$ = number → $F(b) - F(a)$



Indefinite integral $\int f(x)\,dx$ = collection of functions

$F(x) + C$



ex. $\int \sec x \tan x \, dx = \sec x + C$

BECAUSE $\frac{d}{dx}\left[\sec x\right] = \sec x \tan x$

$$\left(\frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{\cos x\,(0) - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \checkmark \right)$$

DERIVATIVE RULES $\rightsquigarrow$ INTEGRATION RULES :

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

ex. FIND $\int \frac{x^2 + 3x - 1}{\sqrt{x}} + \sec^2(2x)\,dx$

$$= \int \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} - \frac{1}{x^{1/2}} + \sec^2(2x)\,dx$$

$$= \int x^{3/2} + 3x^{1/2} - x^{-1/2} + \sec^2(2x)\,dx$$

$$= \int x^{3/2}\,dx + 3\int x^{1/2}\,dx - \int x^{-1/2}\,dx + \int \sec^2(2x)\,dx$$

$$= \frac{1}{\frac{3}{2}+1}x^{\frac{3}{2}+1} + C_1 + 3\,\frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C_2 - \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + C_3 + \underbrace{\frac{1}{2}\tan(2x) + C_4}$$

$$\frac{d}{dx}\left[\frac{1}{2}\tan(2x)\right] = \frac{1}{\cancel{2}}\cancel{2}\sec^2(2x)$$

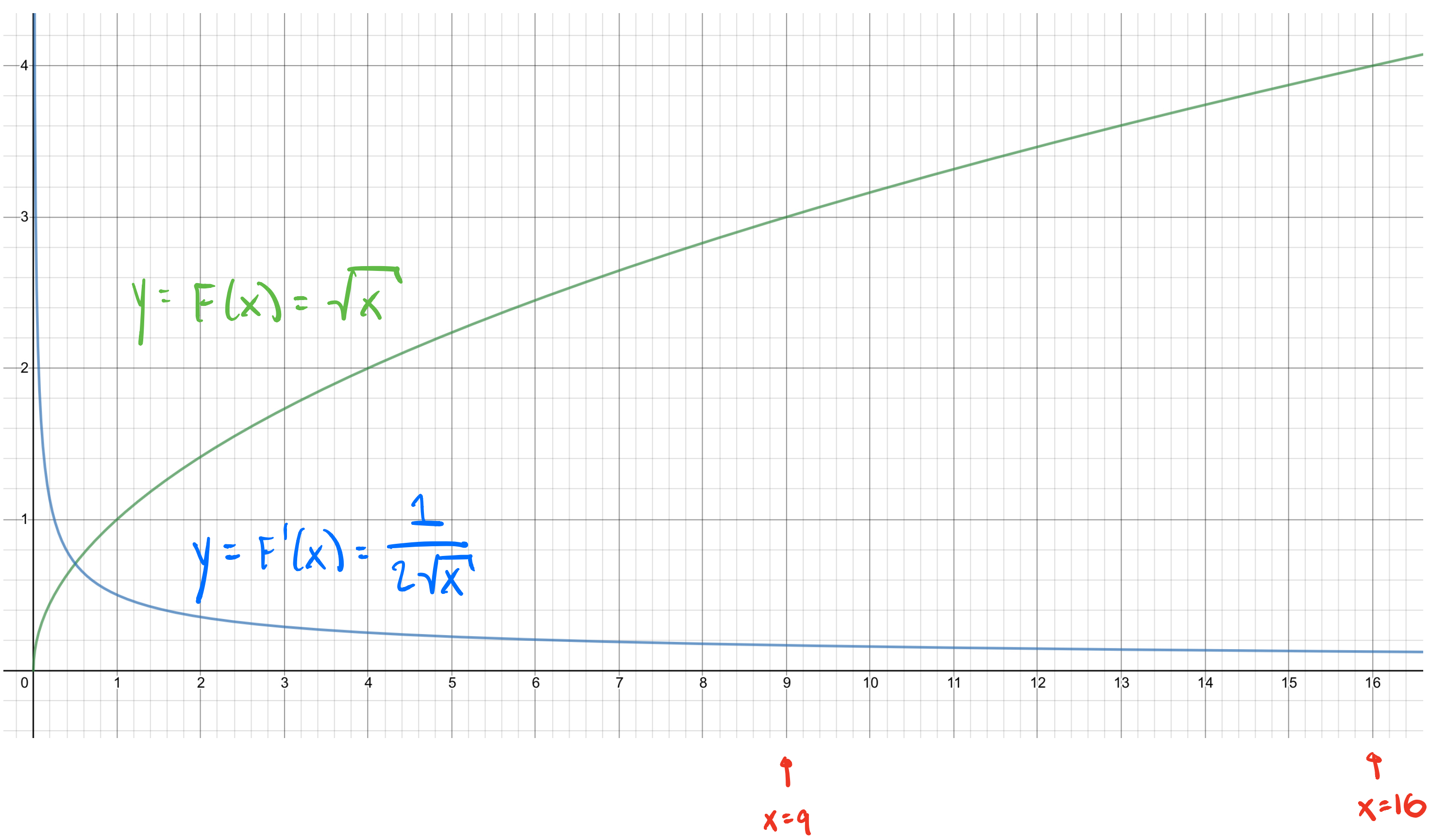
$$= \frac{2}{5}x^{5/2} + 2x^{3/2} - 2x^{1/2} + \frac{1}{2}\tan(2x) + \underbrace{C_1 + C_2 + C_3 + C_4}$$

$$= \frac{2}{5}x^{5/2} + 2x^{3/2} - 2x^{1/2} + \frac{1}{2}\tan(2x) + C$$

SUM OF 4 UNKNOWN #'S IS ONE UNKNOWN #

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$



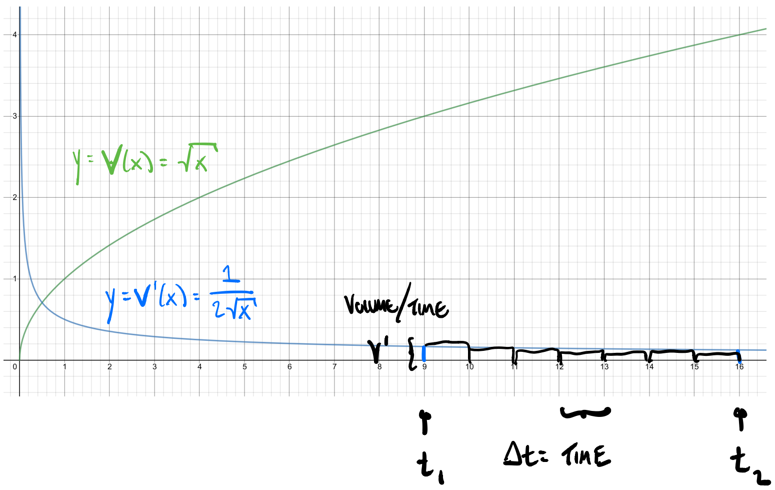
- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

$V'(t) > 0$ FLOW IN

$V'(t) < 0$ FLOW OUT



$$\int_{t_1}^{t_2} V'(t)\,dt$$

AREA = (VOLUME/TIME)(TIME)

= VOLUME

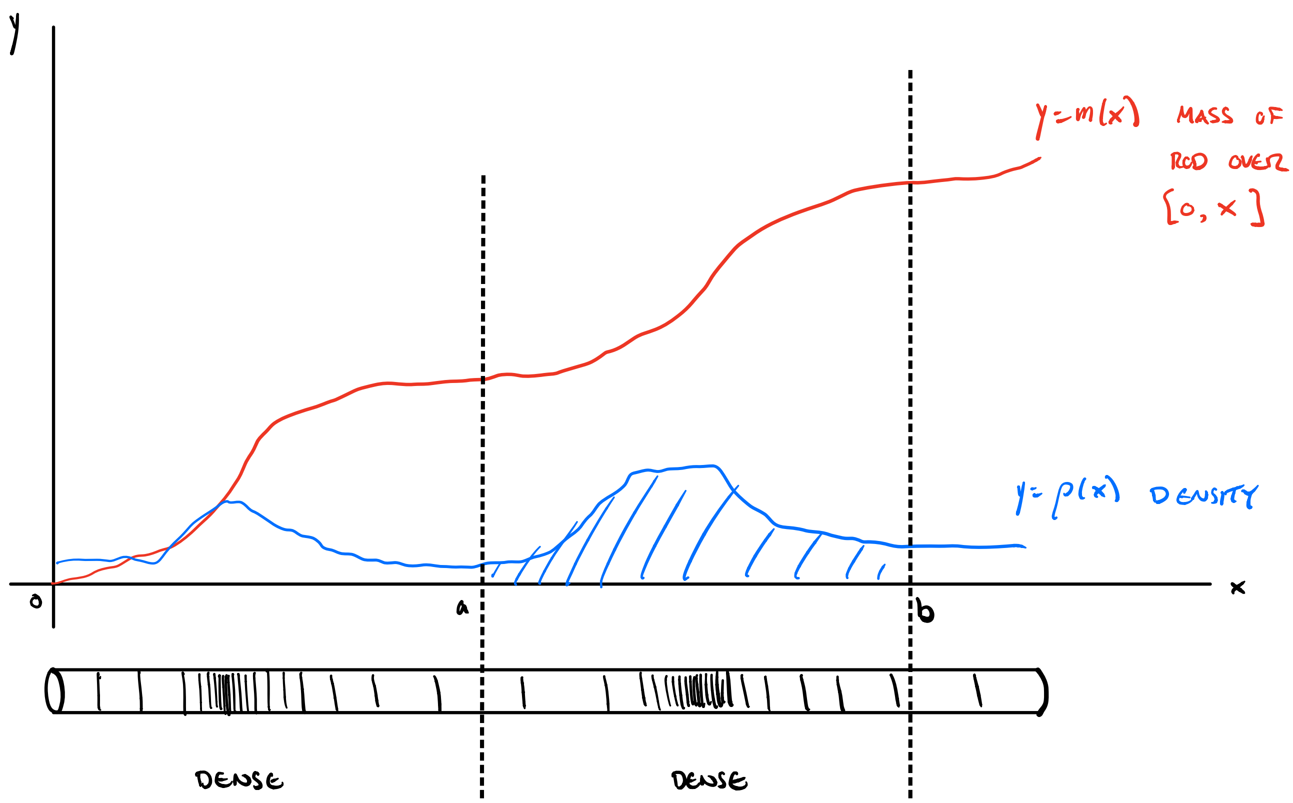
- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

NOTE: HERE WE REFER TO 1D DENSITY

e.g. lbs/ft, kg/m



- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\, dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

2 $$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2. In Section 4.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geqslant 0$ (the particle moves to the right) and also the intervals when $v(t) \leqslant 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

3 * $$\int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\, dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\, dt = A_1 + A_2 + A_3$$

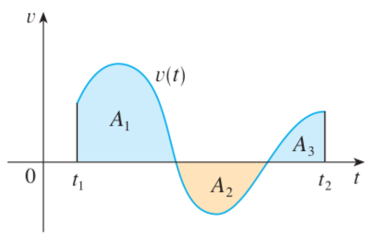


FIGURE 3

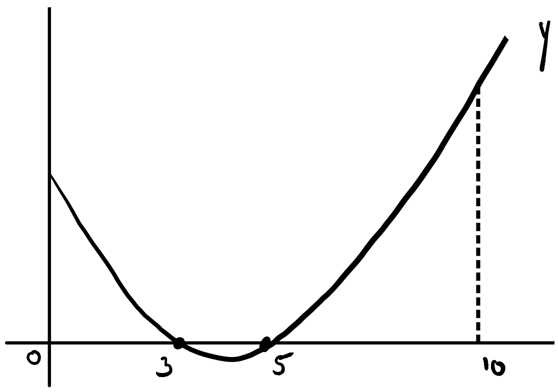
* SPEED = ABSOLUTE VALUE OF VELOCITY.

ex. SUPPOSE THE VELOCITY OF AN OBJECT MOVING IS ALONG A STRAIGHT PATH IS GIVEN BY $v(t) = t^2 - 8t + 15$

(a) FIND THE DISPLACEMENT OF THE OBJECT FROM $t = 0$ TO $t = 10$.

(b) FIND THE DISTANCE TRAVELLED BY THE OBJECT FROM $t = 0$ TO $t = 10$.

(a)



$y = v(t) = t^2 - 8t + 15 = (t-3)(t-5)$

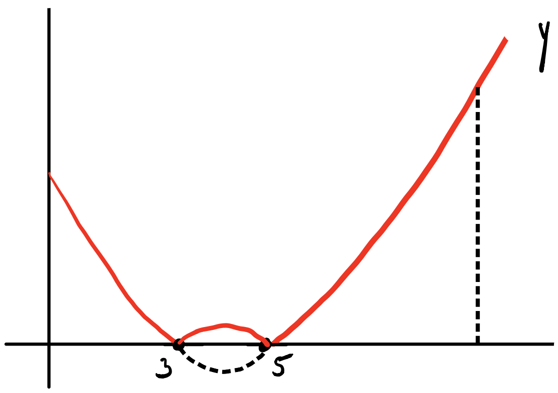
DISPLACEMENT: $\displaystyle\int_0^{10} v(t)\, dt$

$$= \int_0^{10} t^2 - 8t + 15\, dt = \frac{1}{3}t^3 - 4t^2 + 15t \Big|_0^{10}$$

$$= \frac{1}{3}(10)^3 - 4(10)^2 + 15(10)$$

$s(10) - s(0)$ $\quad = 83\frac{1}{3}$.

(b)



$y = |v(t)| = |t^2 - 8t + 15| = |(t-3)(t-5)|$

$$= \begin{cases} t^2 - 8t + 15 & \text{IF } x \le 3 \\ -t^2 + 8t - 15 & \text{IF } 3 \le x \le 5 \\ t^2 - 8t + 15 & \text{IF } 5 \le x \end{cases}$$

DISTANCE TRAVELLED: $\displaystyle\int_0^3 t^2 - 8t + 15\, dt + \int_3^5 -t^2 + 8t - 15\, dt + \int_5^{10} t^2 - 8t + 15\, dt$

$$= \left[\frac{1}{3}t^3 - 4t^2 + 15t\right]_0^3 + \left[-\frac{1}{3}t^3 + 4t^2 - 15t\right]_3^5 + \left[\frac{1}{3}t^3 - 4t^2 + 15t\right]_5^{10}$$

$$= \frac{1}{3}(3)^3 - 4(3)^2 + 15(3) +$$

50. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

$$100 + \underbrace{n(15) - n(0)}_{\text{NET CHANGE IN POPULATION OVER WEEKS 0-15}}$$

ASSUME $n(0) = 100$

48. The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 2.7.3.) What does $\int_a^b I(t)\,dt$ represent?

$\hookrightarrow Q(b) - Q(a)$

NET CHANGE IN CHARGE BETWEEN $t = a$ & $t = b$

§ 4.5 THE SUBSTITUTION RULE

RECALL: THE CHAIN RULE: $\frac{d}{dx}\left[f(g(x)) \right] = f'(g(x))\, g'(x)$

SO NOW: $\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$

$$\int_a^b f'(g(x))\, g'(x)\, dx = f(g(b)) - f(g(a))$$

__ex.__ $\int 2x \sin(x^2)\, dx = \int \sin(x^2) \cdot 2x\, dx$

LET $f'(x) = \sin x \qquad g(x) = x^2 \qquad g'(x) = 2x$

THEN $\int \sin(x^2) \cdot 2x\, dx = -\cos(x^2) + C$

$= \int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$

CHECK: $\frac{d}{dx}\left[-\cos(x^2) \right] = \sin(x^2) \frac{d}{dx}\left[x^2 \right] = \sin(x^2)\, 2x$ ✓

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

INTRODUCE A NEW VARIABLE u FOR THE INNER FUNCTION $g(x)$

① U-SUBSTITUTION

LET $u = g(x)$

DIFFERENTIALS

THEN $\frac{du}{dx} = g'(x) \rightarrow du = g'(x)\, dx$

TRANSLATE TO u: ② REWRITE WITH u, du (NO x's)

$$\rightarrow \int f'(u)\, du = f(u) + C$$

③ INTEGRATE

TRANSLATE BACK TO x:

$$\rightarrow f(g(x)) + C$$

④ BACK-SUBSTITUTE TO x.

ex. $\int x\sqrt{x^2+1}\, dx$

LET $u = x^2+1$ $\quad \frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{1}{2}\, du = x\, dx$

$$\int \sqrt{x^2+1}\; x\, dx = \int \sqrt{u} \cdot \frac{1}{2}\, du$$

$$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\,\frac{2}{3}\,u^{3/2} + C = \frac{1}{3}u^{3/2} + C$$

$$\longrightarrow \boxed{\frac{1}{3}\left(x^2+1\right)^{3/2} + C}$$

CHECK: $\frac{d}{dx}\left[\frac{1}{3}\left(x^2+1\right)^{3/2} + C\right]$

$$= \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}\left(x^2+1\right)^{1/2}\cdot\cancel{2}x = x\sqrt{x^2+1} \quad \checkmark$$