Question Details SCalc8 3.7.060. [3354073]

If $C(x) = 11000 + 600x - 3.2x^2 + 0.004x^3$ is the cost function and $p(x) = 3000 - 8x$ is the demand function, find the production level that will maximize profit. (Hint: If the profit is maximized, then the marginal revenue equals the marginal cost.)

200 units

PROFIT P, PRICE p, COST C, REVENUE R

x = # UNITS SOLD.

$$R(x) = x\,p(x) = x(3000 - 8x) = 3000x - 8x^2$$

$$P(x) = R(x) - C(x)$$

MAXIMIZE: $P'(x) = R'(x) - C'(x) = 0$ FIND CRIT. PNTS.

$$\Rightarrow R'(x) = C'(x)$$

$$3000 - 16x = 600 - 6.4x + .012x^2$$

$$\left(0 = .012x^2 + 9.6x - 2400 \right) \; 1000$$

$$0 = 12x^2 + 9600x - 2400000$$

$$0 = 12(x^2 + 800x - 200000)$$

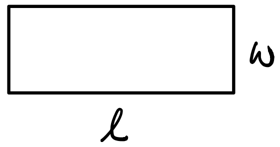
$$0 = 12(x + 1000)(x - 200)$$

$x = -1000$ $\quad$ $x = 200$

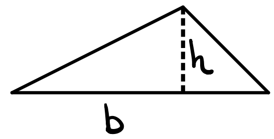
MAX BY 2nd DERIV. TEST.

§4.1 Areas & Distances

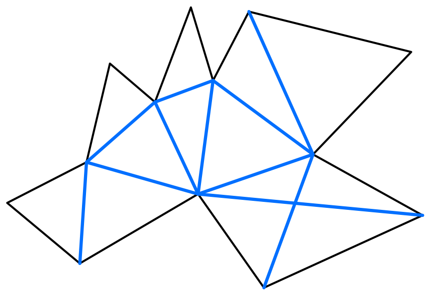
Area



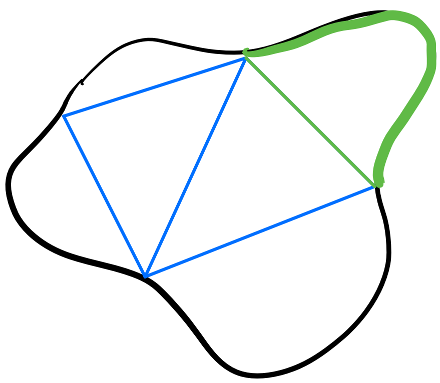
$A = \ell w$



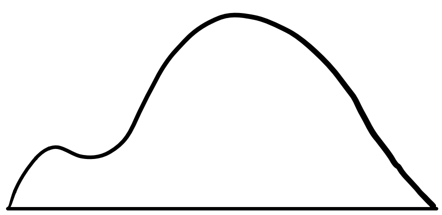
$A = \frac{1}{2} bh$



Any polygon can be triangulated.



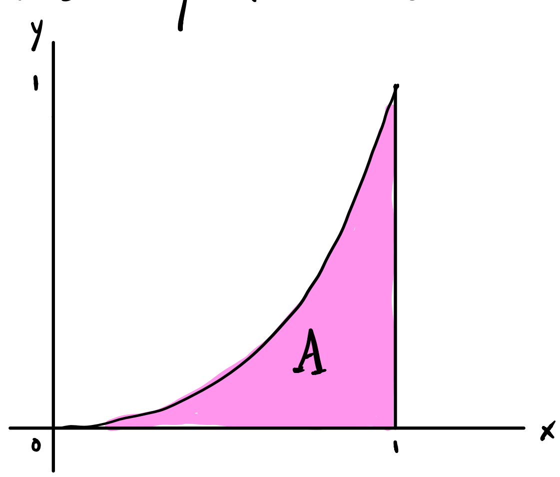
What about shapes with curved edges?

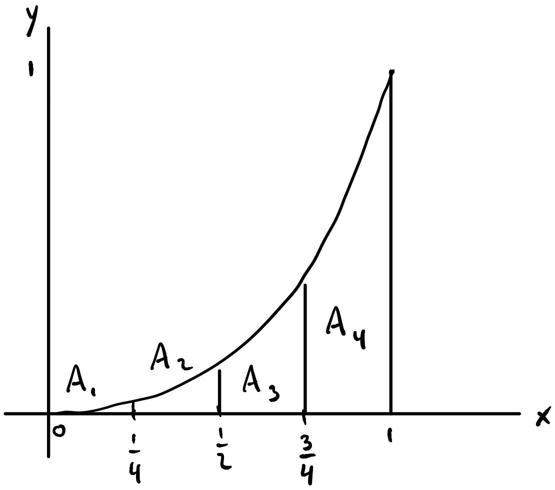


We need a way to calculate the area below a curve.

Let's make this problem precise.

Calculate the area under the curve $y = x^2$ and above the x-axis with $0 \leq x \leq 1$.





IF WE CUT THE INTERVAL $[0,1]$ INTO 4 SUBINTERVALS

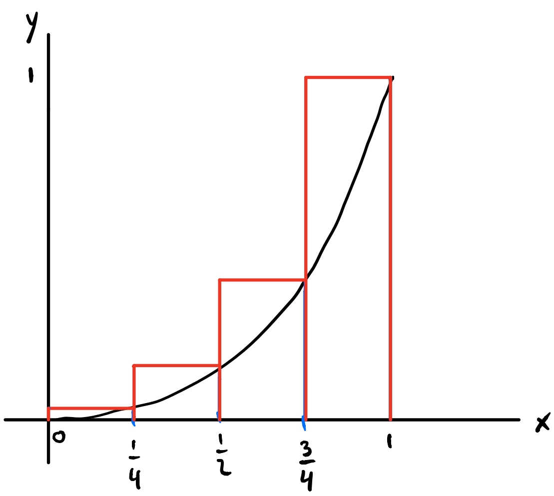
$$I_1 = \left[0, \tfrac{1}{4}\right]$$
$$I_2 = \left[\tfrac{1}{4}, \tfrac{1}{2}\right]$$
$$I_3 = \left[\tfrac{1}{2}, \tfrac{3}{4}\right]$$
$$I_4 = \left[\tfrac{3}{4}, 1\right]$$

THE TOTAL AREA IS THE SUM OF THE AREAS UNDER THE CURVE ABOVE EACH SUBINTERVAL.

$$A = A_1 + A_2 + A_3 + A_4$$

WE CAN APPROXIMATE THE AREA A_i OF EACH VERTICAL STRIP AS THE AREA OF A RECTANGLE WITH BASE $\frac{1}{4}$

AND HEIGHT EQUAL TO THE HEIGHT OF THE CURVE AT THE RIGHT ENDPOINT OF EACH SUBINTERVAL



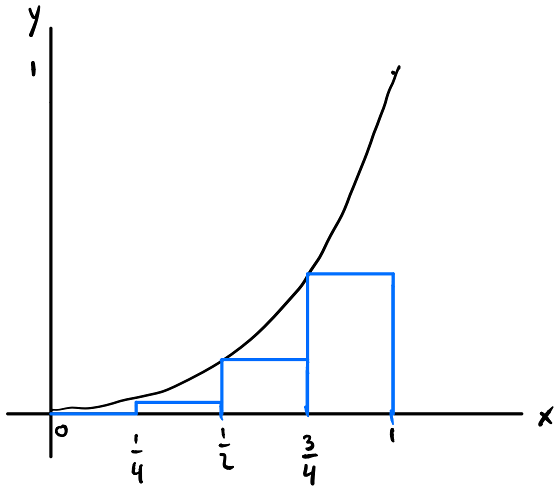
$$A_1 \approx \left(\tfrac{1}{4}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{1}{64}$$
$$A_2 \approx \left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{4}{64}$$
$$A_3 \approx \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{9}{64}$$
$$A_4 \approx (1)^2\left(\tfrac{1}{4}\right) = \tfrac{16}{64}$$

$$\Rightarrow A = A_1 + A_2 + A_3 + A_4$$

$$A \approx \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64} = \frac{30}{64}$$

$A \approx 0.46875$ (OVERESTIMATE) ← CALL THIS R_4

ALTERNATIVELY, WE COULD HAVE APPROXIMATED THE AREA OF EACH VERTICAL STRIP AS A RECTANGLE WITH BASE $\frac{1}{4}$ AND HEIGHT EQUAL TO THE HEIGHT OF THE CURVE AT THE LEFT ENDPOINT OF EACH SUBINTERVAL



$$A_1 \approx (0)^2\left(\tfrac{1}{4}\right) = 0$$

$$A_2 \approx \left(\tfrac{1}{4}\right)^2\left(\tfrac{1}{4}\right) = 1/64$$

$$A_3 \approx \left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{4}\right) = 4/64$$

$$A_4 \approx \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) = 9/64$$

$$\Rightarrow A = A_1 + A_2 + A_3 + A_4$$

$$A \approx 0 + \frac{1}{64} + \frac{4}{64} + \frac{9}{64} = \frac{14}{64}$$

$A \approx 0.21875$ (UNDERESTIMATE) ← CALL THIS L_4

OUR APPROXIMATION BECOMES BETTER WHEN WE MAKE THE NUMBER OF SUBINTERVALS, i.e. THE NUMBER OF RECTANGLES, n LARGER.

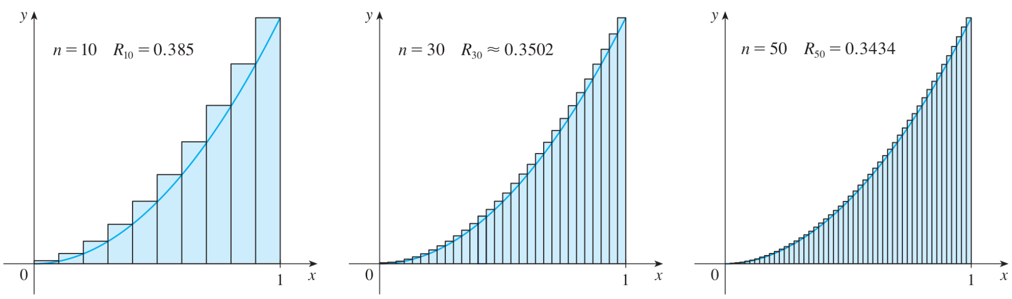


FIGURE 8 Right endpoints produce upper sums because $f(x) = x^2$ is increasing.

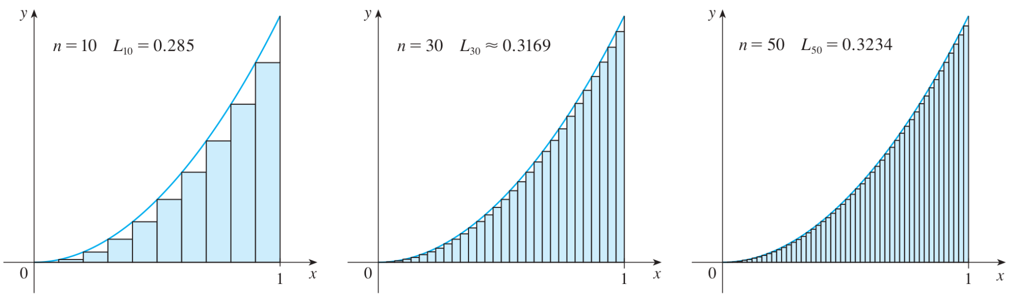


FIGURE 9 Left endpoints produce lower sums because $f(x) = x^2$ is increasing.

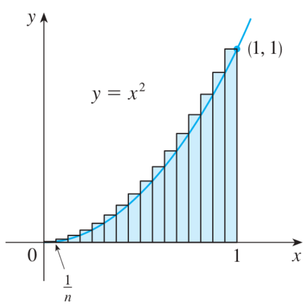
n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

INCREASING UNDERESTIMATES

DECREASING OVERESTIMATES

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \text{AREA}!$$

$$= \frac{1}{3}!$$



PROOF THAT $\lim_{n\to\infty} R_n = \frac{1}{3}$

$$R_n = \left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^2\left(\frac{1}{n}\right) + \left(\frac{3}{n}\right)^2\left(\frac{1}{n}\right) + \dots + \left(\frac{n}{n}\right)^2\left(\frac{1}{n}\right)$$

$$R_n = \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right)$$

$$R_n = \frac{1}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6}$$

$$R_n = \frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}$$

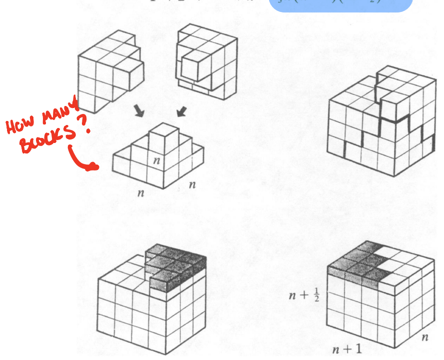
$$R_n = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{3} + \underbrace{\frac{1}{2n}}_{\to 0} + \underbrace{\frac{1}{6n^2}}_{\to 0} = \frac{1}{3} \quad \square$$

Proof without words:
Sum of squares

$1 + 4 + 9 + \dots$

$1^2 + 2^2 + \dots + n^2 = \frac{1}{3}n(n+1)\left(n+\frac{1}{2}\right)$

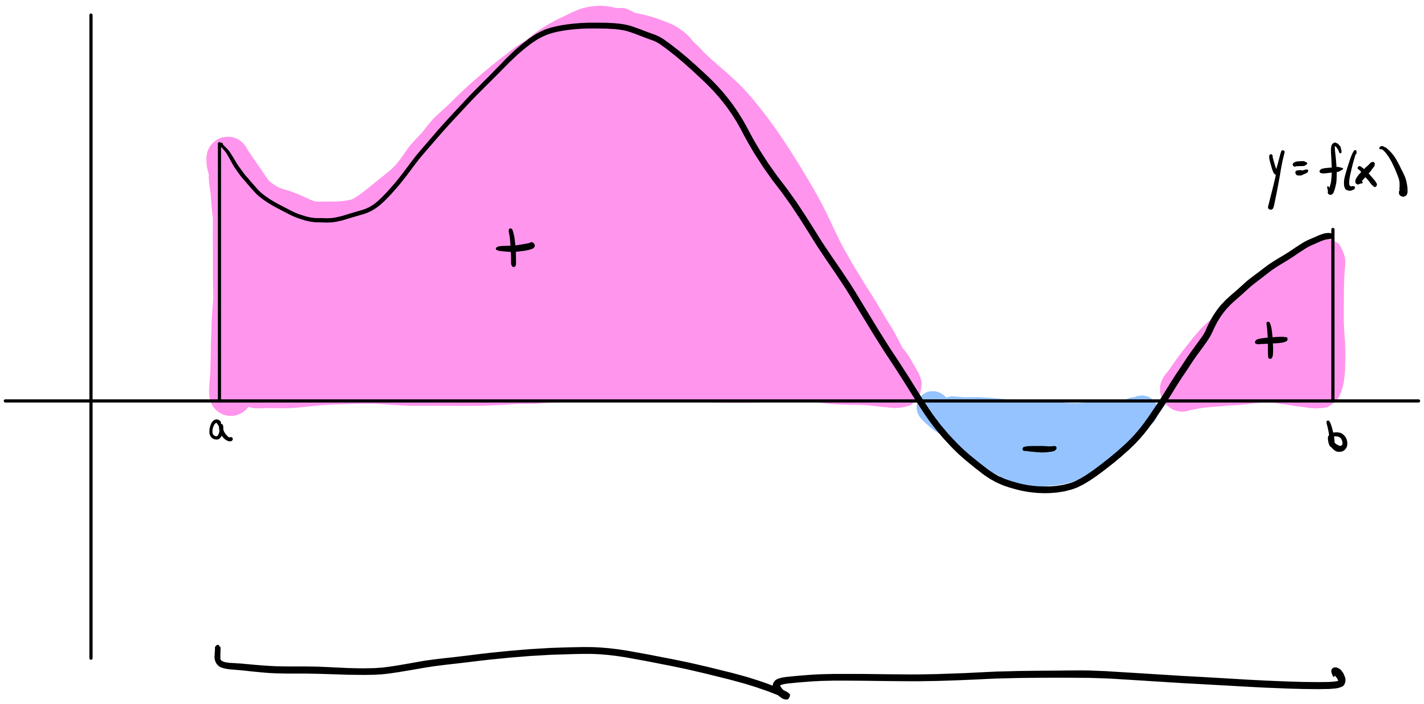


—MAN-KEUNG SIU
University of Hong Kong

$$\left(\frac{1}{3}n(n+1)\left(n+\frac{1}{2}\right) = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}\right)$$

The proof that $\lim_{n \to \infty} L_n = \frac{1}{3}$ is similar.

More generally, we can calculate the signed area A of a region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$.

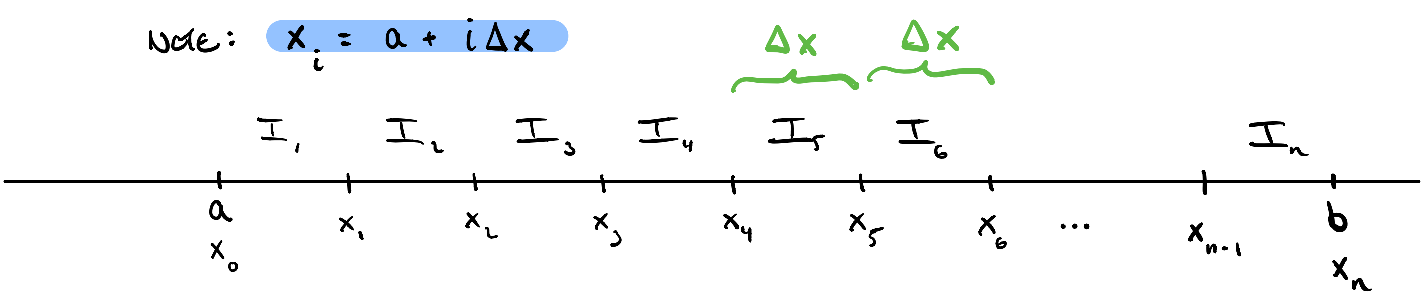


$I = [a, b]$

① Split $I = [a,b]$ into n subintervals of equal length

$\Delta x = \dfrac{b-a}{n}$ with endpoints $a = x_0, x_1, x_2, \ldots, b = x_n$.

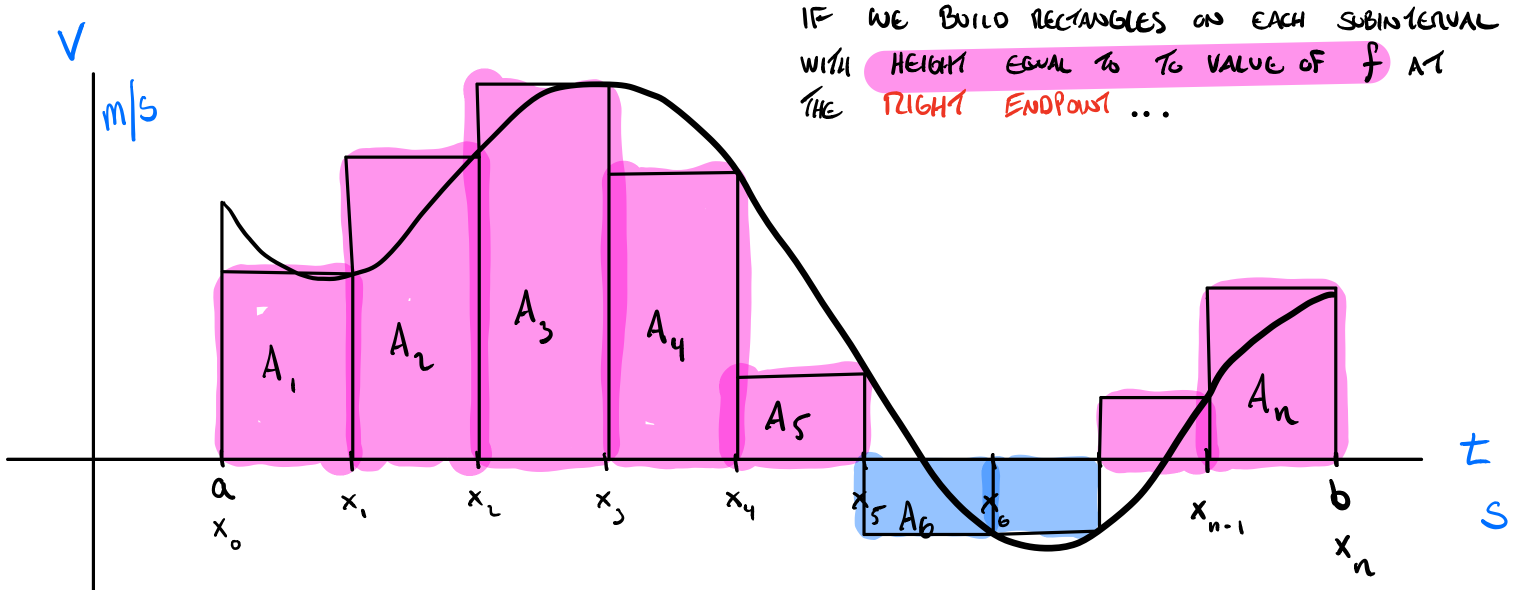
Note: $x_i = a + i\Delta x$



Note: The RIGHT endpoint of subinterval I_i is x_i

The LEFT endpoint of subinterval I_i is x_{i-1}

IF WE BUILD RECTANGLES ON EACH SUBINTERVAL WITH HEIGHT EQUAL TO TO VALUE OF f AT THE RIGHT ENDPOINT ...

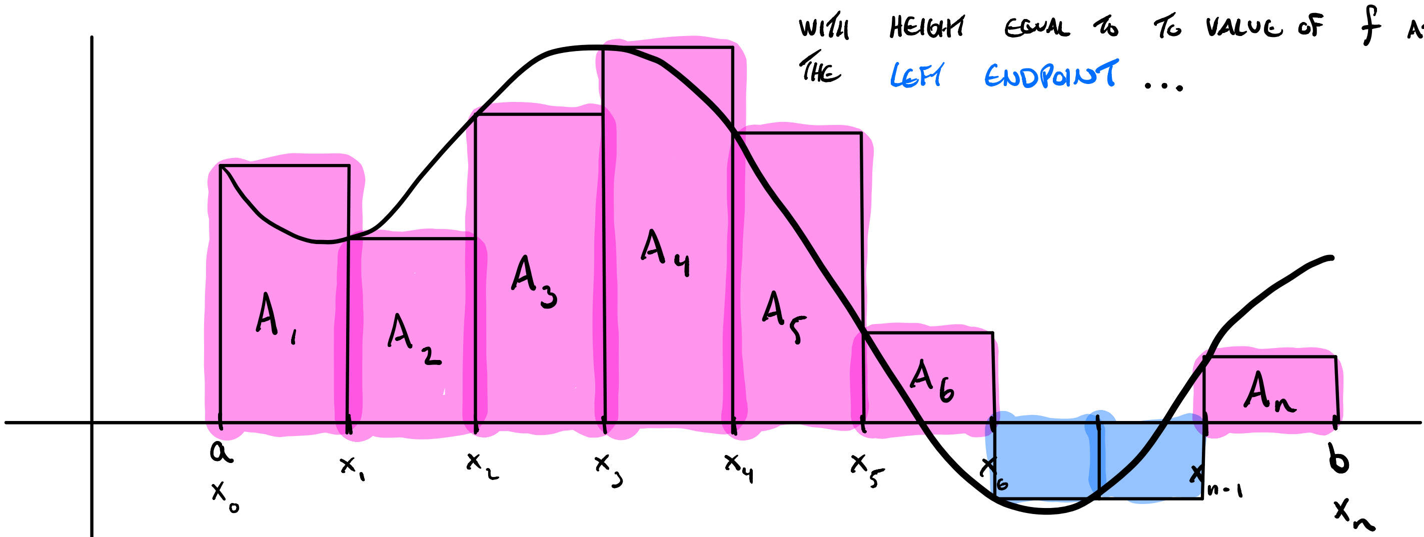


SINGED AREA $A \approx R_n = A_1 + A_2 + \dots + A_n$

$= f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$

$= \sum_{i=1}^{n} f(x_i)\Delta x$ SIGMA NOTATION

IF WE BUILD RECTANGLES ON EACH SUBINTERVAL WITH HEIGHT EQUAL TO TO VALUE OF f AT THE LEFT ENDPOINT ...



SINGED AREA $A \approx L_n = A_1 + A_2 + \dots + A_n$

$= f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$

$= \sum_{i=0}^{n-1} f(x_i)\Delta x$

Sigma Notation:

End value for index

Template for each term

Greek "S" Sum

$$\sum_{i=3}^{6} \frac{2^{i}}{3i+1}$$

Index (variable)

Start value for index

$$\frac{2^3}{3(3)+1} + \frac{2^4}{3(4)+1} + \frac{2^5}{3(5)+1} + \frac{2^6}{3(6)+1}$$

Index increases by one

$$= \frac{8}{10} + \frac{16}{13} + \frac{32}{16} + \frac{64}{19}$$

$$\sum_{i=1}^{3} 2i$$

$2 + 4 + 6$

$$\sum_{i=1}^{3} (2i+1)$$

$3 + 5 + 7$

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

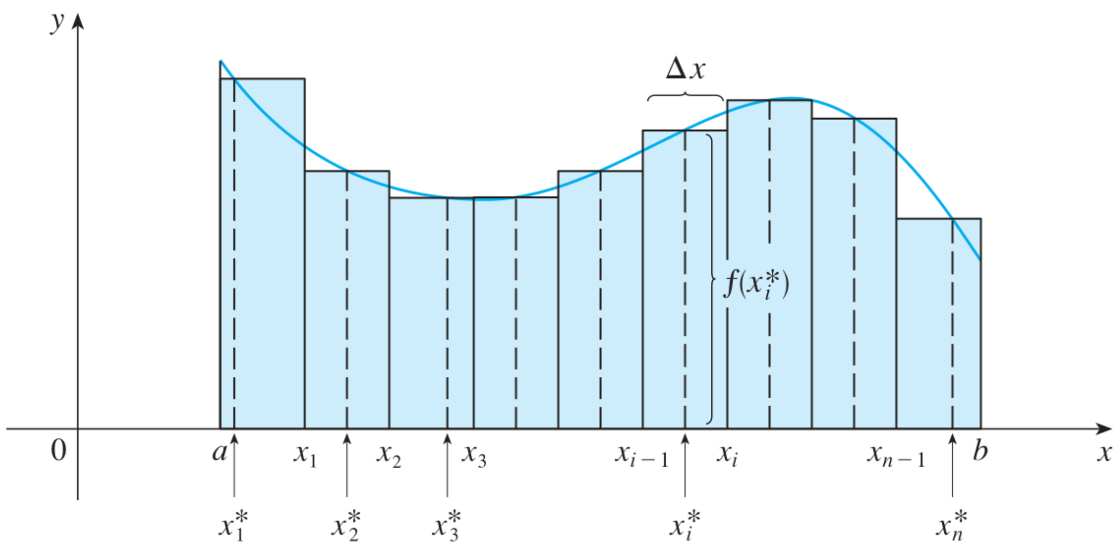
$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

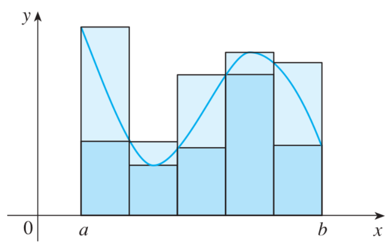
NOTE: $\lim_{n \to \infty} \Delta x = \lim_{n \to \infty} \frac{b-a}{n} = 0$

$$A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

IN FACT: LET x_i^* BE ANY POINT IN $I_i = [x_{i-1}, x_i]$ (SAMPLE POINT)

$$A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$





← SPECIAL CASES:

UPPER SUM: x_i^* = ABS MAX OF f OVER $[x_{i-1}, x_i]$

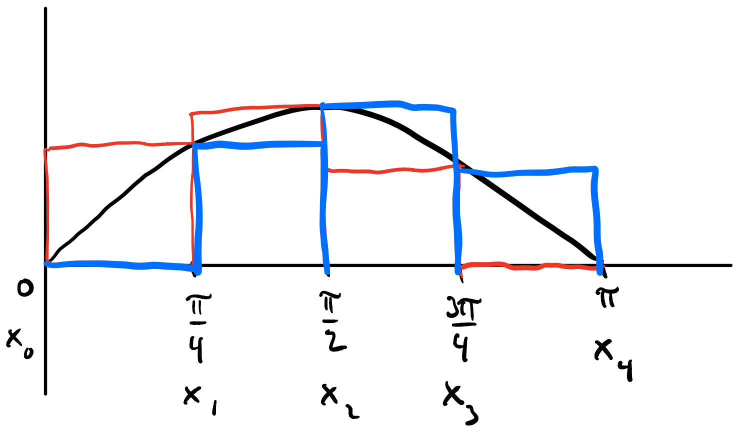
LOWER SUM: x_i^* = ABS MIN OF f OVER $[x_{i-1}, x_i]$

NOTE It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.*

ex. Estimate the area under $y = \sin x$ & above $y=0$ between $x=0$ & $x=\pi$ using (a) R_4

(b) L_4

(c) Give the exact area as a limit.



$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

(a) $R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$

$$= f\left(\frac{\pi}{4}\right)\frac{\pi}{4} + f\left(\frac{\pi}{2}\right)\frac{\pi}{4} + f\left(\frac{3\pi}{4}\right)\frac{\pi}{4} + f(\pi)\frac{\pi}{4}$$

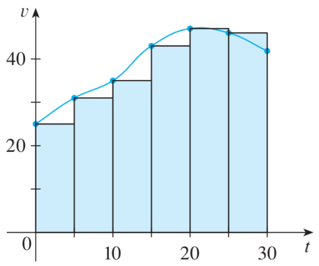
$$= \frac{\pi}{4}\left(\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right)$$

$$= \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0\right)$$

(b) $L_4 = \sum_{i=0}^{3} f(x_i)\Delta x$

$$= \frac{\pi}{4}\left(\sin(0) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{4}\right)\right) =$$

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:



25 31 35 43 47 45

31 35 43 47 45 41

5 SEC 5 SEC 5 SEC 5 SEC 5 SEC 5 SEC

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

$$\text{VELOCITY} = \frac{\text{DISTANCE}}{\text{TIME}} \implies \text{DISTANCE} = \text{VELOCITY} \times \text{TIME}$$

FROM $t=0$ TO $t=5$, DISTANCE $= (25 \text{ ft/s})(5 \text{ s}) = 125 \text{ ft}$

FROM $t=5$ TO $t=10$, DISTANCE $= (31 \text{ ft/s})(5 \text{ s}) = 155 \text{ ft}$

FROM $t=10$ TO $t=15$, DISTANCE $= (35 \text{ ft/s})(5 \text{ s}) = 175 \text{ ft}$

FROM $t=15$ TO $t=20$, DISTANCE $= (43 \text{ ft/s})(5 \text{ s}) = 215 \text{ ft}$

FROM $t=20$ TO $t=25$, DISTANCE $= (47 \text{ ft/s})(5 \text{ s}) = 235 \text{ ft}$

FROM $t=25$ TO $t=30$, DISTANCE $= (45 \text{ ft/s})(5 \text{ s}) = 225 \text{ ft}$ +

TOTAL $= 1130$ ft

(GEOMETRICALLY, THIS IS THE AREA UNDER THE VELOCITY CURVE!)