Find the critical numbers of the function. (Enter your answers as a comma-separated list. Use n to denote any arbitrary integer values. If an answer does not exist, enter DNE.)

$g(\theta) = 36\theta - 9\tan(\theta)$

$\theta =$ $\boxed{2\pi n + \frac{\pi}{3}, 2\pi n + \frac{2\pi}{3}, 2\pi n + \frac{4\pi}{3}, 2\pi n + \frac{5\pi}{3}}$

$$g(\theta) = 32\theta - 8\tan\theta$$

$$g'(\theta) = 32 - 8\sec^2\theta = 0$$

$$32 = 8\sec^2\theta$$

$$4 = \sec^2\theta$$

$$4 = \frac{1}{\cos^2\theta} \Rightarrow \cos^2\theta = \frac{1}{4}$$

$$g'(\theta) = 32 - 8\left(\frac{1}{\cos\theta}\right)^2 \quad \text{UNDEFINED}$$

WHEN $\cos\theta = 0$

$$\left.\begin{aligned} \theta &= \frac{\pi}{2} + n2\pi, \\ &-\frac{\pi}{2} + n2\pi \end{aligned}\right\} \text{NOT IN } \mathrm{DOM}(g)$$

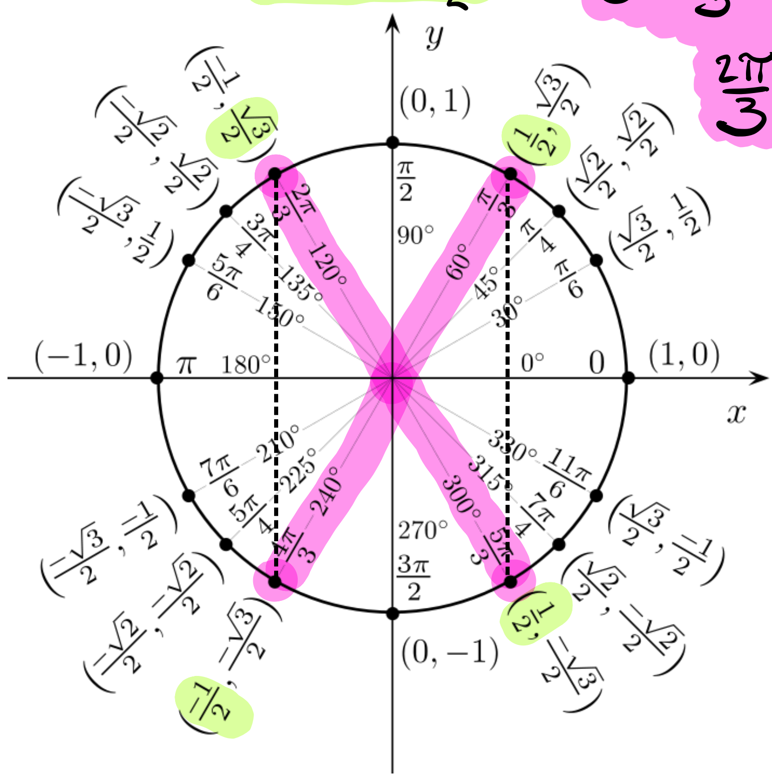
$\otimes$

$$\cos\theta = \pm\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} + n\pi, \ \frac{2\pi}{3} + n\pi$$

Def: c IS CRITICAL # OF f IF

$c \in \mathrm{Dom}(f)$ & EITHER

- $f'(c) = 0$, OR
- $f'(c)$ IS UNDEFINED.



$$x = \cos\theta, \quad y = \sin\theta$$

§ 3.3 CONTINUED

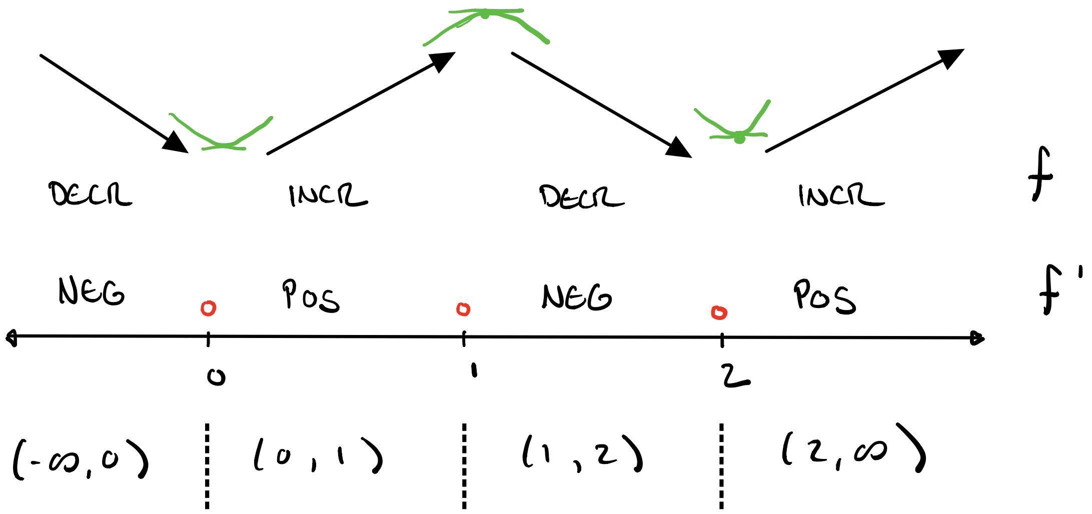
ex. FIND THE INTERVAL(S) ON WHICH

$$f(x) = x^4 - 4x^3 + 4x^2$$

IS INCREASING / DECREASING.

{ FIND THE INTERVAL(S) ON WHICH $f'(x)$ IS POS / NEG.

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-2)(x-1) = 0$$



X = 0 IS LOCAL MIN

X = 1 IS LOCAL MAX

X = 2 IS LOCAL MIN

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

ex. FIND ALL LOCAL MAX/MIN'S OF

$$f(x) = x^{2/3}(6-x)^{1/3}$$

DOM$(f) = \mathbb{R}$

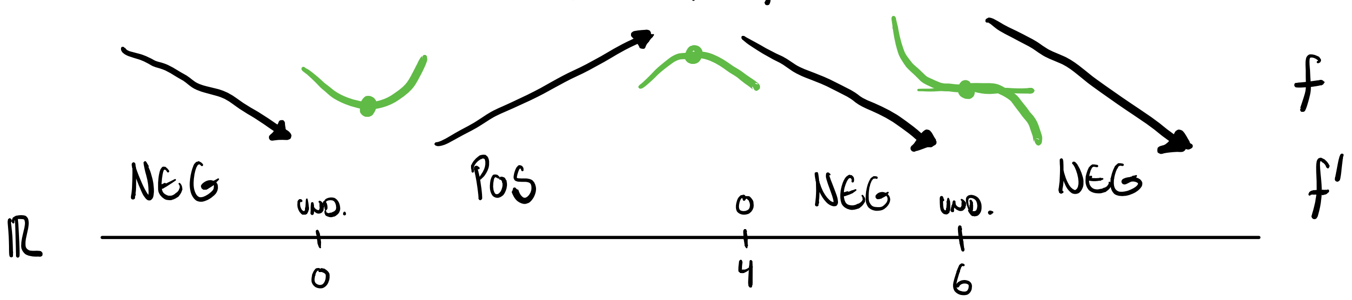
FIND CRITICAL POINTS: $f'(x) = \frac{2}{3}x^{-1/3}(6-x)^{1/3} - x^{2/3}\,\frac{1}{3}(6-x)^{-2/3}$

$$f'(x) = \underbrace{\frac{1}{3}x^{-1/3}(6-x)^{-2/3}}_{}\Big[2(6-x) - x\Big]$$

GCD = PROD. OF COMMON FACTORS RAISED TO LOWEST EXPONENT

$$f'(x) = \frac{12-3x}{3x^{1/3}(6-x)^{2/3}} = \frac{\cancel{3}(4-x)}{\cancel{3}x^{1/3}(6-x)^{2/3}}$$

CRITICAL #'S : $\{\ 6,\ 4,\ 0\ \}$
(UND.) (0) (UND)



$(4-x)$	POS	POS	NEG	NEG
$x^{1/3}$	NEG	POS	POS	POS
$(6-x)^{2/3}$	POS	POS	POS	POS

$\uparrow$

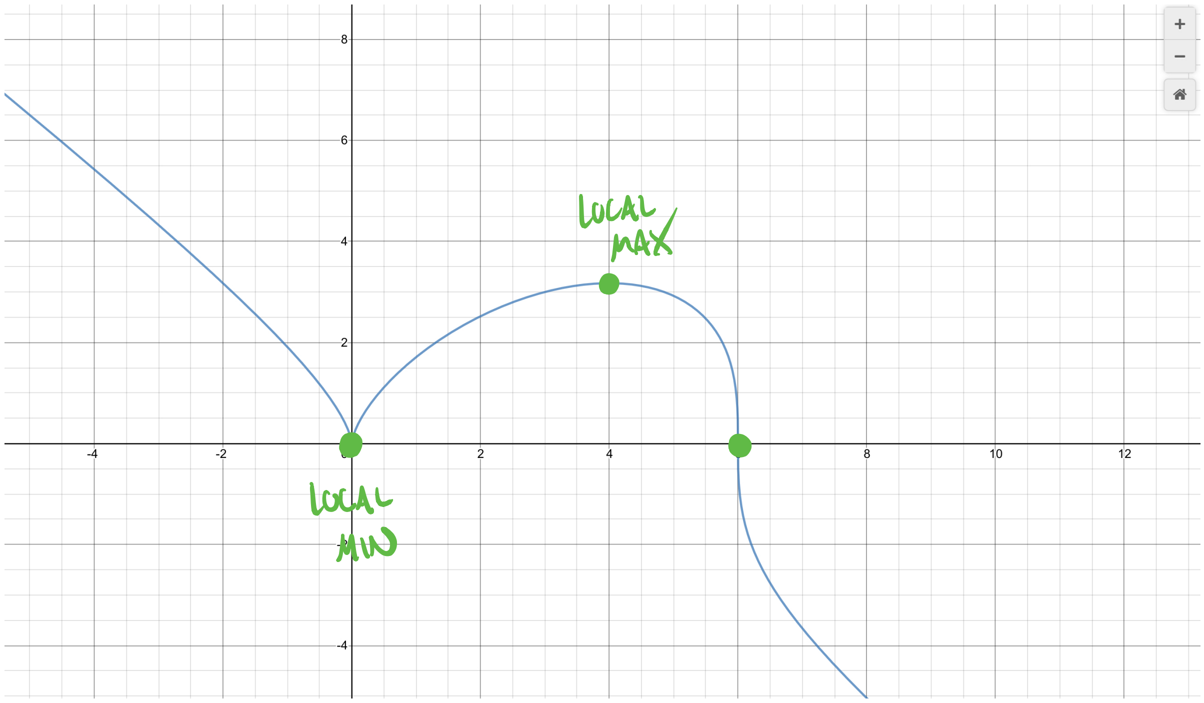
$\left[(6-x)^{1/3}\right]^2$

LOCAL MAX @ $x = 4$, LOCAL MAX VALUE $f(4) = 4^{2/3}\cdot 2^{1/3}$
$= (2^2)^{2/3}\, 2^{1/3}$
$= 2^{5/3}$

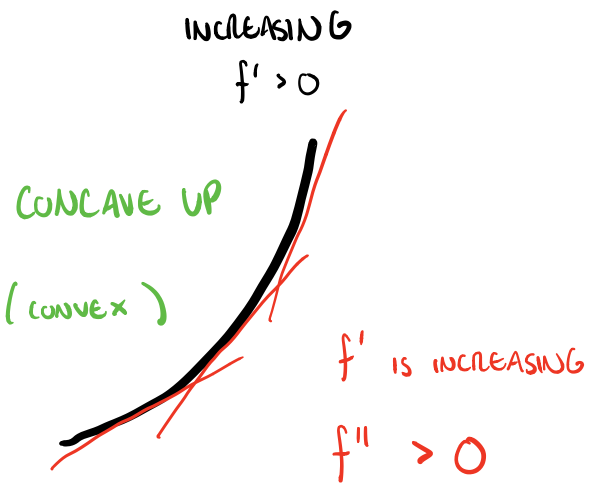
LOCAL MIN @ $x = 0$,

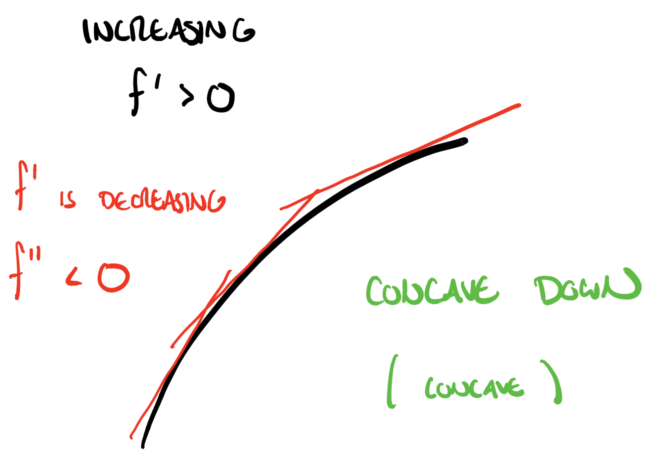
(x = 6 IS NEITHER)

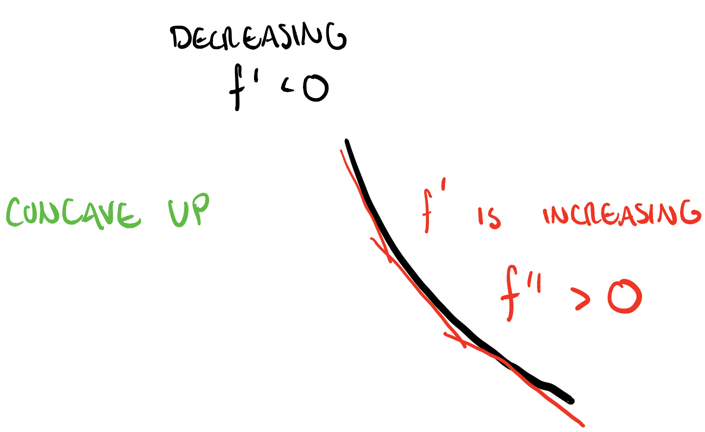
LOCA MIN VALUE $f(0) = 0$

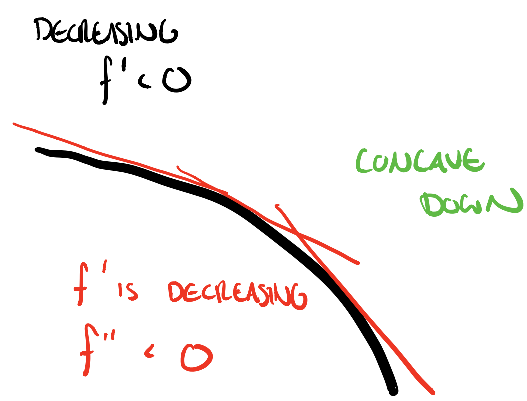


Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.





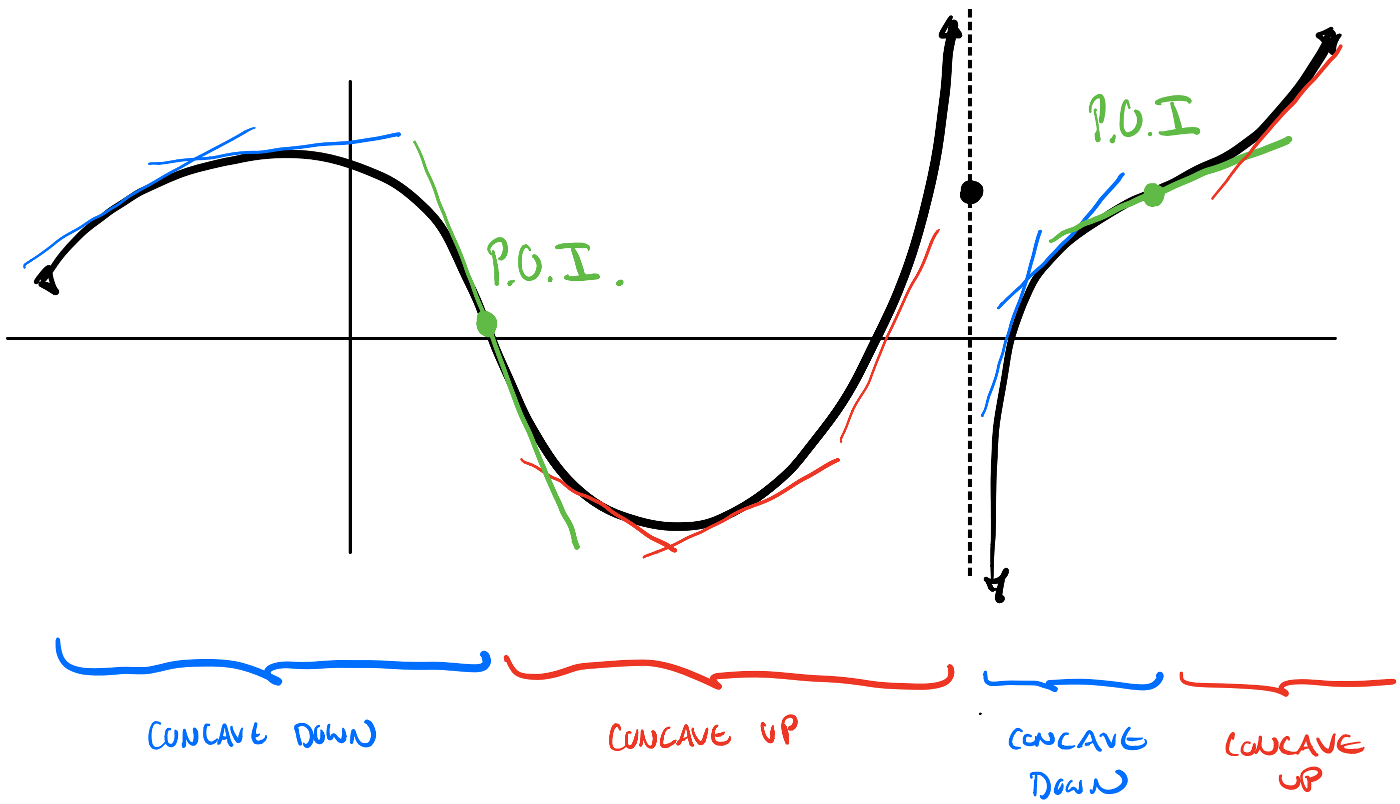




Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

POINT OF INFLECTION (P.O.I.)

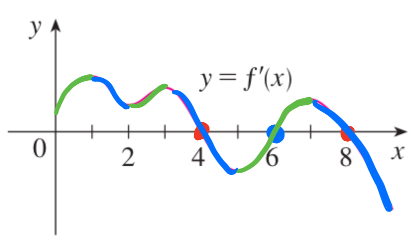
8. The graph of the first derivative f' of a function f is shown.

(a) On what intervals is f increasing? Explain.

(b) At what values of x does f have a local maximum or minimum? Explain.

(c) On what intervals is f concave upward or concave downward? Explain.

(d) What are the x-coordinates of the inflection points of f? Why?



(a) = f' POSITIVE

$(0,4) \cup (6,8)$

OPEN INTERVALS ALWAYS

(b) @ $x=4$, $x=8$

f' CHANGES FROM ⊕ TO ⊖

f CHANGES FROM ↗ TO ↘

LOCAL MAX

@ $x=6$ LOCAL MIN

f' : NEG TO POS

f : ↘ TO ↗

(c) f IS CONCAVE UP : f' IS INCREASING

$(0,1) \cup (2,3) \cup (5,7)$

f IS CONCAVE DOWN : f' IS DECREASING

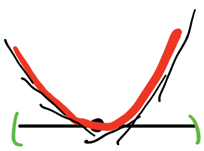
$(1,2) \cup (3,5) \cup (7,9)$

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

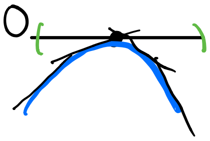
(a) $f'(c)=0$



$f''(c) > 0 \iff f'$ IS INCREASING ON I

HENCE NEG TO POS $\rightarrow$ LOCAL MIN

(b) $f'(c)=0$



$f''(c) < 0 \iff f'$ IS DECREASING ON I

HENCE POS TO NEG — LOCAL MIN

9–14

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity and the inflection points.

9. $f(x) = x^3 - 3x^2 - 9x + 4$

10. $f(x) = 2x^3 - 9x^2 + 12x - 3$

11. $f(x) = x^4 - 2x^2 + 3$

12. $f(x) = \dfrac{x}{x^2 + 1}$

13. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

14. $f(x) = \cos^2 x - 2 \sin x, \quad 0 \leqslant x \leqslant 2\pi$

15–17 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

15. $f(x) = 1 + 3x^2 - 2x^3$

16. $f(x) = \dfrac{x^2}{x - 1}$

17. $f(x) = \sqrt{x} - \sqrt[4]{x}$

$\text{Dom}(f) = \{ x \in \mathbb{R} \mid x \neq 1 \}$

$2x^2 - 2x - x^2 \qquad x^2 - 2x$

16. CRITICAL #'S : $f'(x) = \dfrac{(x-1)\cdot 2x - x^2(1)}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$

CRITICAL #'S $= \{ 0, 2 \}$

FIRST DERIV. TEST

		0		1		2	
f	↗		↘		↘		↗
f'	POS	0	NEG	NOT IN DOM	NEG	0	POS
x	NEG		POS		POS		POS
$x-2$	NEG		NEG		NEG		POS
$(x-1)^2$	POS		POS		POS		POS

LOCAL MAX @ $x=0$ LOCAL MIN @ $x=2$

LOCAL MAX VALUE $f(0)=\frac{0^2}{0-1}=0$ LOCAL MIN VALUE $f(2)=\frac{2^2}{2-1}=4$

SECOND DERIVATIVE TEST $f'(x)=\frac{x^2-2x}{(x-1)^2}$

$$f''(x)=\frac{(x-1)^2(2x-2)-(x^2-2x)\cdot 2(x-1)}{(x-1)^4}$$

$$=\frac{(x-1)\left[\overbrace{(x-1)(2x-2)-(x^2-2x)\cdot 2}^{2x^2-2x-2x+2-(2x^2-4x)}\right]}{(x-1)^4}$$

$$=\frac{2+4x}{(x-1)^3}=\frac{2(1+2x)}{(x-1)^3}$$

PLUG IN CRIT #'S INTO $f''(x)$

$f''(0)=\frac{2}{-1}<0 \implies x=0$ LOCAL MAX

$f''(2)=\frac{10}{1}>0 \implies x=2$ LOCAL MIN