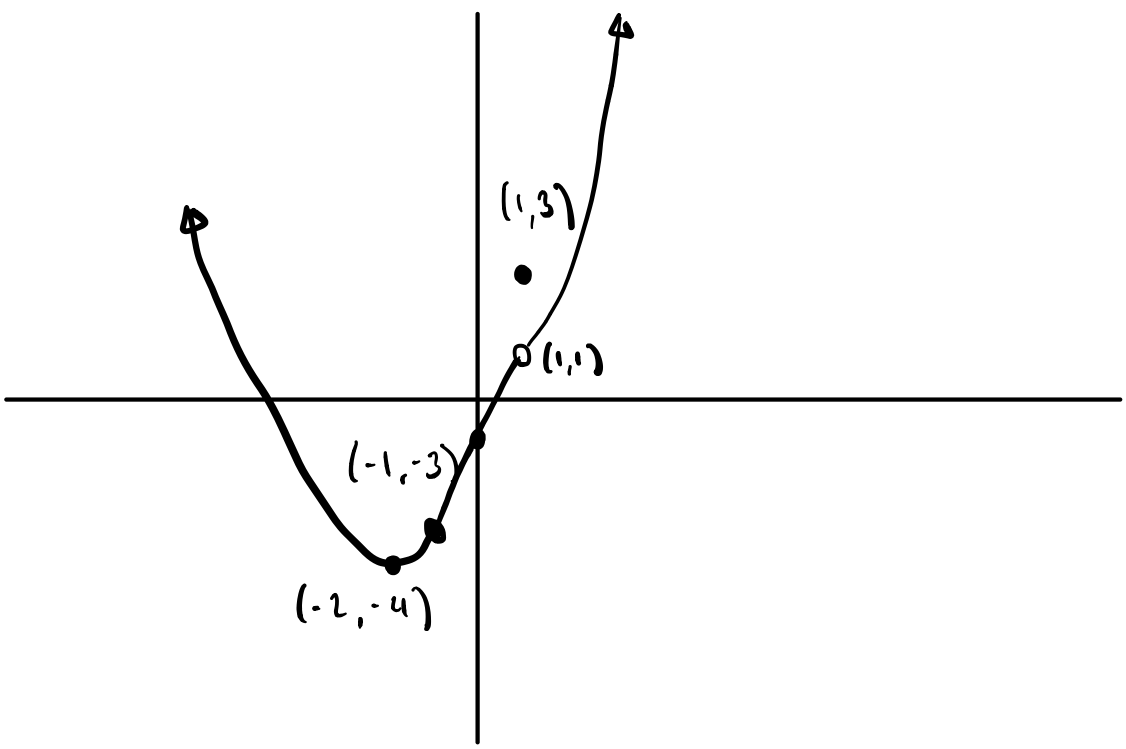
REVIEW FOR 1206 MIDTERM EXAM

1. CONSIDER THE FUNCTION

$$f(x) = \begin{cases} x^2 & \text{IF } 1 < x \\ 3 & \text{IF } x = 1 \\ 2x - 1 & \text{IF } -1 \leq x < 1 \\ (x+2)^2 - 4 & \text{IF } x < -1 \end{cases}$$

(a.) SKETCH THE GRAPH $y = f(x)$.



(b) FIND $\lim_{x \to 1} f(x)$ OR SHOW IT D.N.E.

SINCE f FOLLOWS DIFFERENT RULES FOR $x<1$ & $x>1$, WE MUST CALCULATE THE ONE-SIDED LIMITS SEPARATELY.

$\lim_{x\to 1} f(x)$ EXIST IF & ONLY IF

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \quad \text{(BOTH EXIST)}$$

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 2x-1 = 2(1)-1 = 1 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x^2 = (1)^2 = 1 \end{aligned}\right\} \text{EXIST + EQUAL}$$

$$\therefore \quad \lim_{x\to 1} f(x) = 1$$

1c) IS f CONTINUOUS AT $x = 1$?

MEANS

$$\underbrace{\lim_{x\to 1} f(x)}_{1} \neq \underbrace{f(1)}_{3}$$

1. LIMIT EXIST ✓ 1
2. $1 \in \mathrm{Dom}(f)$ ✓ 3
3. EQUAL ✗

No.

(d) SHOW THAT f IS CONTINUOUS AT $x = -1$

(DEF. OF CONTINUOUS)

(e) USE DEFINITION OF DERIVATIVE AS A LIMIT TO SHOW THAT f IS DIFFERENTIABLE AT $x = -1$.

IF f IS DIFFERENTIABLE AT $x = -1$ THEN f IS CONTINUOUS AT $x = -1$.

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \begin{cases} \lim\limits_{h \to 0^+} \dfrac{\left(2(-1+h)-1\right) - (-3)}{h} \\ \lim\limits_{h \to 0^-} \dfrac{\left((-1+h)+2\right)^2 - 4 - (-3)}{h} \end{cases}$$

$$= \begin{cases} \lim\limits_{h \to 0^+} \dfrac{-2+2h-1+3}{h} = \lim\limits_{h \to 0^+} \dfrac{2h}{h} = 2 \\ \lim\limits_{h \to 0^-} \dfrac{(1+h)^2 - 1}{h} = \lim\limits_{h \to 0^-} \dfrac{\cancel{1}+2h+h^2-\cancel{1}}{h} = \lim\limits_{h \to 0^-} \dfrac{\cancel{h}(2+h)}{\cancel{h}} = 2 \end{cases}$$

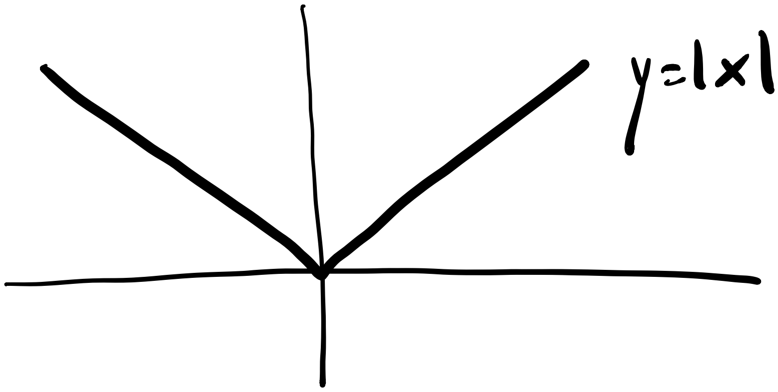
$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = 2$$

$\therefore f$ IS DIFFERENTIABLE AT $x = -1$.

(d) $f'(-1)$ EXISTS

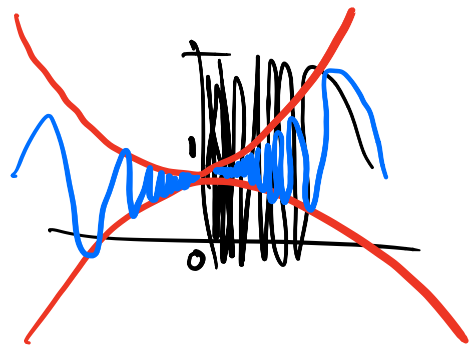
$\Rightarrow f$ IS DIFF. AT -1

$\Rightarrow f$ IS CONTINUOUS AT -1.



CONTINUOUS AT $x=0$

NOT DIFFERENTIABLE AT $x=0$.



2. USE SQUEEZE THM TO CALCULATE

$$\lim_{x \to 0} 1 + x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right)$$

$$-1 \le \sin\theta \le 1 \quad , \quad \forall\, \theta \in \mathbb{R}$$ FOR ALL

$$-1 \le \sin\frac{2\pi}{\sqrt{|x|}} \le 1 \quad , \quad x \neq 0$$

$$-x^2 \le x^2 \sin\frac{2\pi}{\sqrt{|x|}} \le x^2 \quad , \quad x \neq 0$$

$(x^2 > 0)$

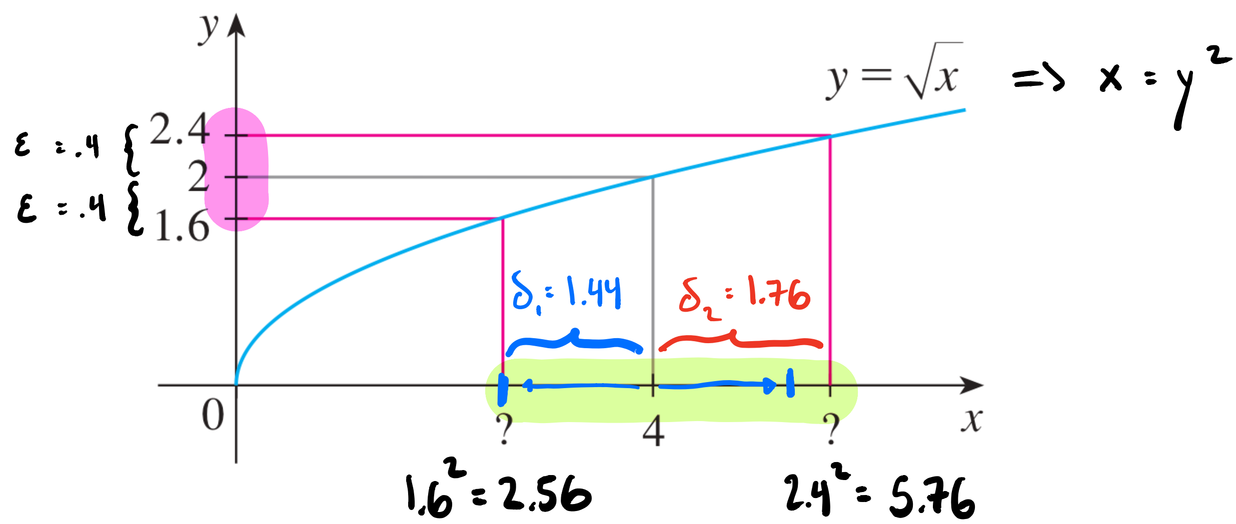
$$1 - x^2 \le 1 + x^2 \sin\frac{2\pi}{\sqrt{|x|}} \le \quad , \quad \forall\, x \neq 0$$

$$\underbrace{\lim_{x\to 0} 1-x^2}_{1} \le \lim_{x\to 0} 1+x^2\sin\frac{2\pi}{\sqrt{|x|}} \le \underbrace{\lim_{x\to 0} 1+x^2}_{1}$$

BY SQUEEZE THM, $\displaystyle\lim_{x\to 0} 1+x^2\sin\frac{2\pi}{\sqrt{|x|}} = 1$.

3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \qquad |x-4| < \delta \qquad \text{then} \qquad |\sqrt{x}-2| < 0.4$$



$$\delta = \min(\delta_1, \delta_2) = \delta_1 = 1.44$$

4. (a.) STATE THE INTERMEDIATE VALUE THEOREM.

> **10** **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

(b.) SHOW THAT THE EQUATION $\dfrac{1}{3-\sqrt{x}} = \cos(\pi x)$

HAS AT LEAST ONE SOLUTION.

EQUIVALENTLY: SHOW THAT $f(x) = \dfrac{1}{3-\sqrt{x}} - \cos(\pi x) = 0$

FOR SOME VALUE x.

SINCE $f(1) = \dfrac{1}{3-\sqrt{1}} - \cos(\pi) = \dfrac{1}{2} - (-1) = \dfrac{3}{2}$

AND $f(16) = \dfrac{1}{3-\sqrt{16}} - \cos(16\pi) = \dfrac{1}{3-4} - 1 = -1-1 = -2$

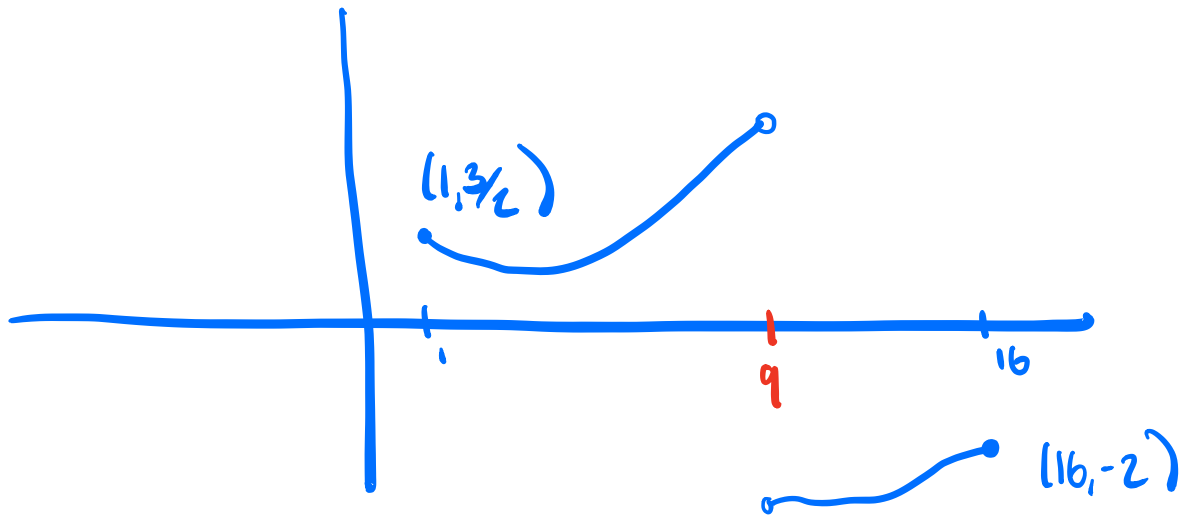
SINCE $f(16) = -2 < 0 < \dfrac{3}{2} = f(1)$

BY I.V.T. $\exists$ A # c : $1 < c < 16$ SUCH THAT

$f(c) = 0$

X

THIS WOULD BE TRUE IF f IS CONTINUOUS ON $[1, 16]$



BUT: f is NOT continuous on [1, 16]

$f(x) = \frac{1}{3-\sqrt{x}} - \cos(\pi x)$ is continuous on its domain

$[0, 9) \cup (9, \infty)$

9

① $x \geq 0$

$3 - \sqrt{x} \neq 0$

$3 \neq \sqrt{x}$

$9 \neq x$

f is continuous on [16, 25]

$f(16) = -2$, $f(25) = \frac{1}{1-\sqrt{25}} - \cos(25\pi)$

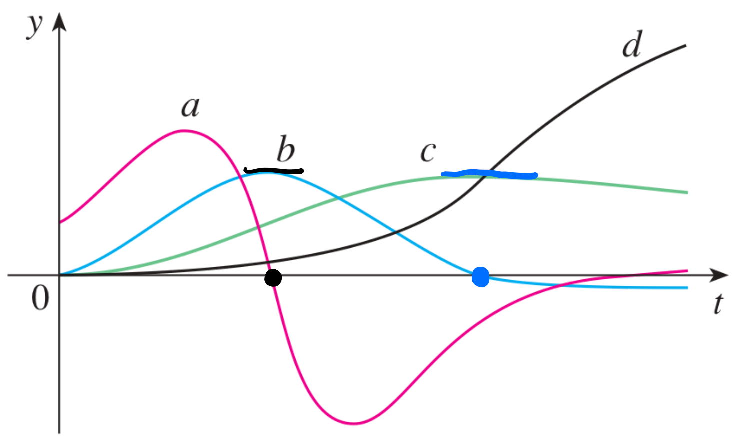
$= -\frac{1}{4} + 1 = \frac{3}{4}$

SINCE $f(16) < 0 < f(25)$

THERE MUST BE A # c, $16 < c < 25$

5.1. $f(c) = 0$.

The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



a IS THE DERIV. OF b — VALUE OF a = 0 WHEN SLOPE OF b = 0

b IS THE DERIV. OF c — VALUE OF b = 0 WHEN SLOPE OF c = 0

c IS THE DERIV. OF d — VALUE OF c IS GREATEST WHEN SLOPE OF d IS GREATEST

6. CALCULATE THE LIMIT OR SHOW THAT IT DOES NOT EXIST.

$$\lim_{x \to 0} \sqrt[3]{\frac{\sin(3x)}{\sin(5x)}}$$

$$= \sqrt[3]{\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}}$$

$$= \sqrt[3]{\lim_{x \to 0} \left(\frac{3 \sin 3x}{3x} \cdot \frac{5x}{5 \sin 5x} \right)}$$

$$= \sqrt[3]{\underbrace{3 \lim_{x \to 0} \frac{\sin 3x}{3x}}_{1} \times \underbrace{\frac{1}{5} \lim_{x \to 0} \frac{5x}{\sin 5x}}_{1}}$$

SIMILARLY

LET $\theta = 3x$

$\theta \to 0$ AS $x \to 0$

$$\lim_{x \to 0} \frac{\sin 3x}{3x} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

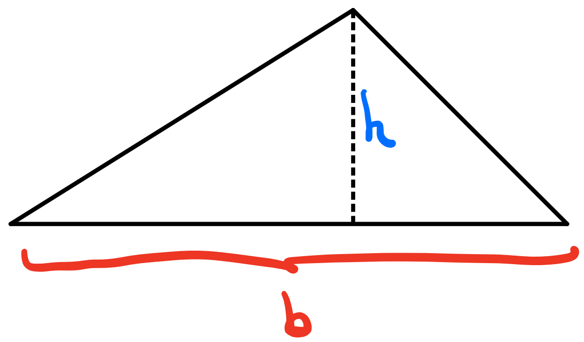
$$= \sqrt[3]{3 \times 1 \times \frac{1}{5} \times 1} = \sqrt[3]{\frac{3}{5}}$$

INCREASING $\Rightarrow$ DERIV. POS.

DECREASING $\Rightarrow$ DERIV. NEG.

9. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

QUANTITIES: $A = \frac{1}{2} bh$



RATES OF CHANGE: $\frac{d}{dt} A = \frac{d}{dt}\left[\frac{1}{2} bh\right]$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt} h + b \frac{dh}{dt}\right)$$

PRODUCT RULE

FIND $\frac{db}{dt}$ WHEN $h = 10$, $A = 100$

$A = \frac{1}{2} bh \Rightarrow b = \frac{2A}{h} = \frac{2(100)}{10} = 20$

GIVEN: $\frac{dh}{dt} = 1$ $\frac{dA}{dt} = 2$

$$2 = \frac{1}{2}\left(\frac{db}{dt}(10) + (20)(1)\right)$$

$$4 = 10\frac{db}{dt} + 20$$

$$-16 = 10\frac{db}{dt}$$

$$-1.6 = \frac{db}{dt}$$