CHAIN RULE → IMPLICIT DIFF

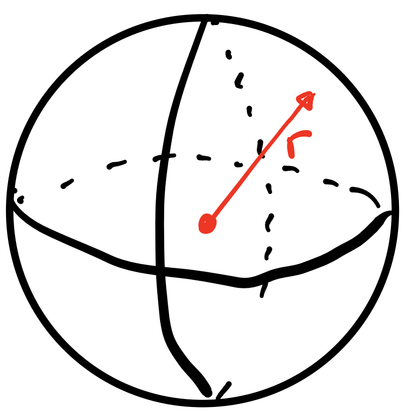
CHAIN RULE → RELATED RATES

§2.8 Related Rates

EX. A SPHERICAL BALLOON IS BEING INFLATED SUCH THAT ITS VOLUME IS INCREASING AT A RATE OF $.25\ m^3/s$. WHEN THE RADIUS OF THE BALLOON IS $1\ m$, FIND THE RATE AT WHICH THE RADIUS IS INCREASING.

(1) DRAW A PICTURE AND INTRODUCE NOTATION.

LET VARIABLES REPRESENT ALL QUANTITIES THAT CHANGE OVER TIME, AND ASSUME ALL THESE VARIABLES ARE DIFFERENTIABLE FUNCTIONS OF t, TIME.



LET r = RADIUS OF SPHERE

V = VOLUME OF SPHERE

$r(t)$

$V(t)$

(2) IDENTIFY WHAT INFORMATION IS GIVEN & WHAT IS BEING ASKED.

GIVEN $\frac{dV}{dt} = .25$ $V'(t)$

FIND $\frac{dr}{dt}$ WHEN $r = 1$ $r'(t)$

(3) WRITE DOWN AN EQUATION THAT RELATES THE VARIABLES.
YOU MAY HAVE TO COMBINE TWO OR MORE EQUATIONS TO GET A SINGLE EQUATION RELATING THE VARIABLE WHOSE RATE YOU WANT TO THE VARIABLE(S) WHOSE RATE(S) YOU KNOW.

* DO NOT PLUG IN CONSTANTS FOR VARIABLES YET!

$$V = \frac{4}{3}\pi r^3$$

(4) DIFFERENTIATE THE EQUATION WITH RESPECT TO t.
SOLVE FOR THE RATE YOU WANT TO FIND.

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \frac{d}{dr}\left[\frac{4}{3}\pi r^3\right] \cdot \frac{dr}{dt}$$

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CHAIN RULE

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

(5) EVALUATE BY PLUGGING IN ALL KNOWN VALUES.

$$\left.\frac{dr}{dt}\right|_{r=1} = \frac{1}{4\pi(1)^2}(.25) = \boxed{\frac{1}{16\pi}\ m/s} \approx .02\ m/s$$

IN SUMMARY:

(1) DRAW A PICTURE AND INTRODUCE NOTATION.
LET VARIABLES REPRESENT ALL QUANTITIES THAT CHANGE OVER TIME, AND ASSUME ALL THESE VARIABLES ARE DIFFERENTIABLE FUNCTIONS OF t, TIME.

(2) IDENTIFY WHAT INFORMATION IS GIVEN & WHAT IS BEING ASKED.

(3) WRITE DOWN AN EQUATION THAT RELATES THE VARIABLES.
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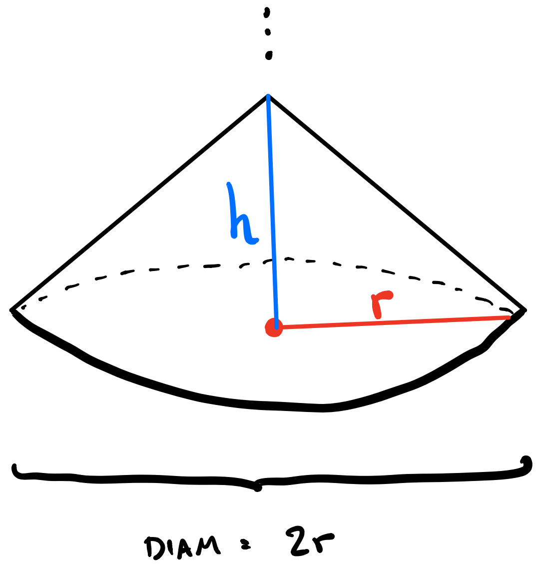
(4) DIFFERENTIATE THE EQUATION WITH RESPECT TO t.
SOLVE FOR THE RATE YOU WANT TO FIND.

(5) EVALUATE BY PLUGGING IN ALL KNOWN VALUES.

A growing sand pile Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.



$$h = \frac{3}{8}(2r) \Rightarrow h = \frac{3}{4} r$$

$$r = \frac{4}{3} h$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h$$

$$V = \frac{16\pi}{27} h^3$$

RELATED QUANTITIES

GIVEN: $\frac{dV}{dt} = 10$

FIND $\frac{dh}{dt}$ WHEN $h=4$.

$$\frac{dV}{dt} = \frac{16\pi}{27} 3h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{27}{48\pi h^2} \frac{dV}{dt}$$

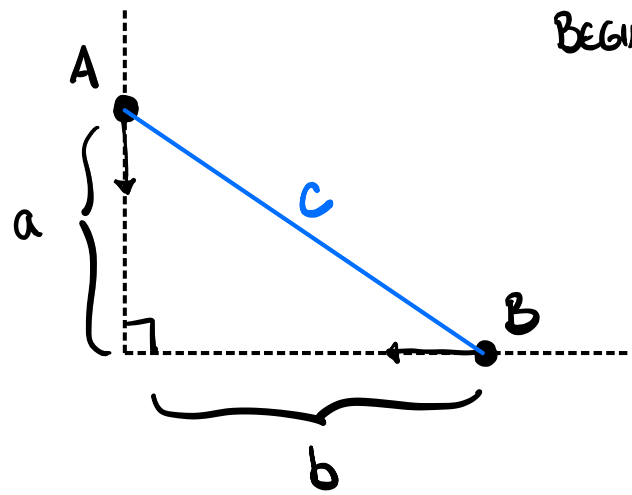
RELATED RATES (DERIVATIVES)

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{27}{48\pi (4\,m)^2} \cdot 10\ m^3/min =$$

$$\frac{270}{768\pi}\ \frac{m^3}{m^2 \cdot min} = \frac{270}{768\pi}\ m/min$$

$$= \frac{2.7}{768\pi}\ cm/min$$

Commercial air traffic Two commercial airplanes are flying at an altitude of 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?



BEGIN BY RELATING QUANTITIES:

$$a^2 + b^2 = c^2$$

$$\Big\downarrow \frac{d}{dt}$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

GIVEN $\frac{da}{dt} = -442$ knots

$\frac{db}{dt} = -481$ knots

FIND $\frac{dc}{dt}$ WHEN $a = 5$

$b = 12$

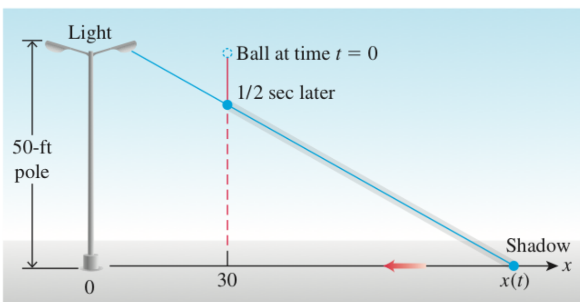
$$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c}$$

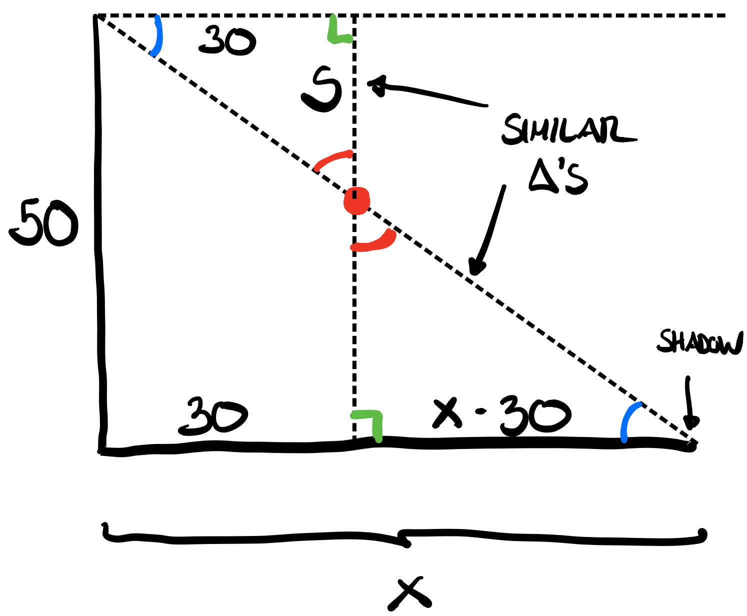
$$\left.\frac{dc}{dt}\right|_{\substack{a=5\\ b=12}} = \frac{5(-442) + 12(-481)}{13} = -614 \text{ KNOTS}$$

$c = \sqrt{5^2+12^2} = 13$

-614 NAUT. MILES / HR

A moving shadow A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)





GIVEN: $s = 16t^2$ ft $\rightarrow \frac{ds}{dt} = 32t$

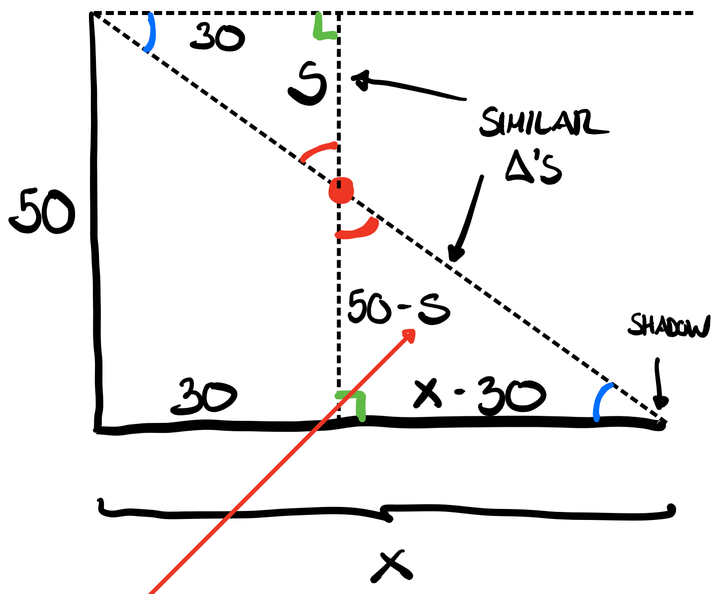
FIND: $\frac{dx}{dt}$ WHEN $t = \frac{1}{2}$ $\rightarrow \left.\frac{ds}{dt}\right|_{t=\frac{1}{2}} = 16$

IN ORDER TO RELATE $\frac{dx}{dt}$ TO $\frac{ds}{dt}$

LET'S FIRST RELATE x TO s.

Similar Δ's:

$$\frac{\text{Height}_1}{\text{Base}_1} = \frac{\text{Height}_2}{\text{Base}_2}$$



Note: In class I accidentally wrote $50 - x$ instead of $50 - s$.

$$\frac{s}{30} = \frac{50-s}{x-30}$$

$$s(x-30) = 30(50-s)$$

$$sx - 30s = 1500 - 30s \Rightarrow sx = 1500$$

$$\frac{d}{dt}[sx] = \frac{d}{dt}[1500]$$

$$\frac{ds}{dt}x + s\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{-x\frac{ds}{dt}}{s}$$

$$\left.\frac{dx}{dt}\right|_{t=\frac{1}{2}} = \frac{-375(16)}{4} = \boxed{-1500 \text{ ft/s}}$$

$$\left(\begin{array}{l} s = 16\left(\frac{1}{2}\right)^2 = 4 \\ x = \frac{1500}{4} = 375 \end{array} \right)$$

$$\frac{ds}{dt} = 32\left(\frac{1}{2}\right) = 16$$