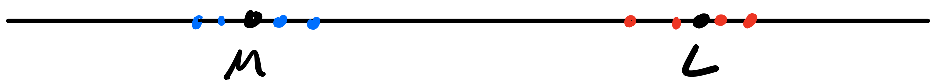
§ 1.6 LIMIT LAWS (CONTINUED)

THM: IF $f(x) \le g(x)$ FOR x NEAR a, $x \ne a$,

AND IF $\lim\limits_{x\to a} f(x)$ AND $\lim\limits_{x\to a} g(x)$ BOTH EXIST

THEN $\underbrace{\lim\limits_{x\to a} f(x)}_{L} \le \underbrace{\lim\limits_{x\to a} g(x)}_{M}$.

ASSUME IF $L > M$



THEN AS $x \to a$, $f(x) \to L$ AND $g(x) \to M$

SO EVENTUALLY, AS $x \to a$, WE MUST HAVE

$f(x) \ge g(x)$

THIS CONTRADICTION IMPLIES THE ASSUMPTION IS WRONG. ✓

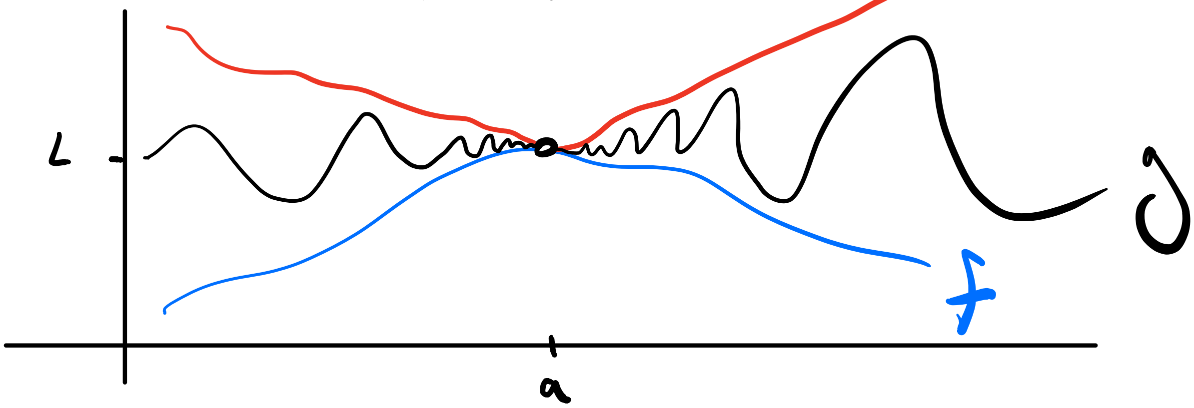
THM (SQUEEZE THM)

IF $f(x) \le g(x) \le h(x)$ FOR x NEAR a, $x \ne a$,

AND IF

$\lim\limits_{x\to a} f(x) = L$ $\lim\limits_{x\to a} h(x) = L$ (SAME LIMIT)

THEN $\lim\limits_{x \to a} g(x) = L$



ex.

39. Prove that $\lim\limits_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

40. Prove that $\lim\limits_{x \to 0^+} \sqrt{x}\,[1 + \sin^2(2\pi/x)] = 0$.

$3 < 5$

$-6 > -10$

NOTE:

$$-1 \le \cos(\theta) \le 1 \quad \text{FOR ANY } \theta \in \mathbb{R}$$

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1 \quad \text{FOR ANY } x \neq 0$$

$$\underset{f}{-x^4} \le \underset{g}{x^4 \cos\left(\frac{2}{x}\right)} \le \underset{h}{x^4} \quad (x \neq 0,\ x^4 > 0)$$

APPLY SQUEEZE THM.

$$\lim_{x \to 0} -x^4 \le \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) \le \lim_{x \to 0} x^4$$

$$-\left(\lim_{x \to 0} x\right)^4 \le \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) \le \left(\lim_{x \to 0} x\right)^4$$

$$0 \le \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) \le 0$$

$$\therefore \lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) = 0$$

ex. $\displaystyle \lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right]$

$$\left(\begin{array}{l} \text{As } x\to 0^+, \quad \frac{1}{x} \to \infty \\ \qquad\qquad\quad\;, \quad \frac{2\pi}{x} \to \infty \end{array}\right)$$

$$\left(\lim_{x\to 0^+} \sin\left(\frac{2\pi}{x}\right) \text{ DNE}\right)$$

Note: $\quad -1 \le \sin\left(\frac{2\pi}{x}\right) \le 1 \quad , \; x\neq 0$

$$0 \le \sin^2\left(\frac{2\pi}{x}\right) \le 1$$

$$1 \le 1+\sin^2\left(\frac{2\pi}{x}\right) \le 2$$

$$\sqrt{x} \le \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] \le 2\sqrt{x} \qquad (\sqrt{x} > 0)$$

$$\lim_{x\to 0^+} \sqrt{x} \le \lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] \le \lim_{x\to 0} 2\sqrt{x}$$

$$0 \le \lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] \le 0$$

$$\therefore \lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] = 0.$$

§1.7 Precise Def. of a Limit (Rec. 9/15)

§1.8 Continuity

- Def of continuity
- Types of discontinuities
- Properties: applying limit laws to continuous functions
- Compositions of continuous functions
- Intermediate value thm.

Def: f is continuous at a if

$$\underbrace{\underbrace{\lim_{x \to a} f(x)}_{(1)\ \text{exists}} = \underbrace{f(a)}_{(2)\ \text{defined},\ a \in \mathrm{Dom}(f)}}_{(3)\ \text{equal}}$$

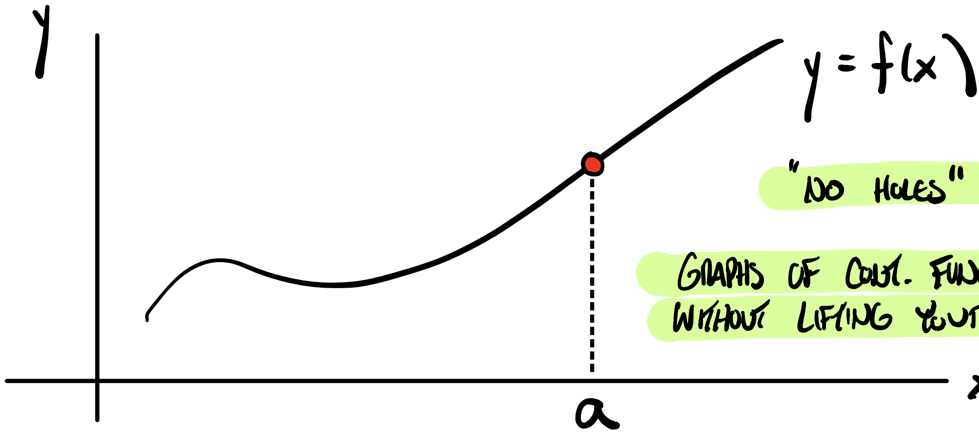
(application)

$$C = 2\pi r \rightarrow r = \frac{1}{2\pi} C$$

$$r(C) = \frac{1}{2\pi} C \leftarrow \text{continuous function.}$$

$$\lim_{C \to a} r(C) = \lim_{C \to a} \frac{1}{2\pi} C$$

$$= \frac{1}{2\pi} \lim_{C \to a} C = \frac{1}{2\pi} a = r(a)$$

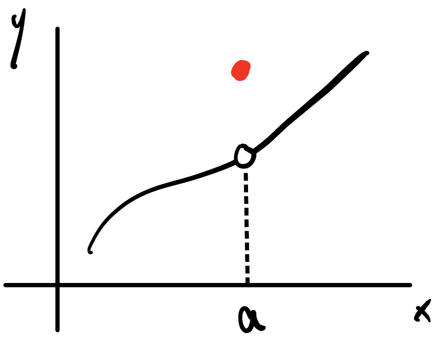


"NO HOLES" "NO BREAKS"

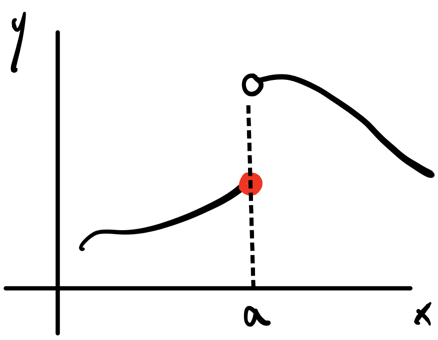
GRAPHS OF CONT. FUNC'S CAN BE DRAWN WITHOUT LIFTING YOUR PENCIL

f IS **CONTINUOUS ON AN INTERVAL** I IF f IS CONTINUOUS AT EVERY POINT IN I.

TYPES OF DISCONTINUITIES: (POINT-BY-POINT)



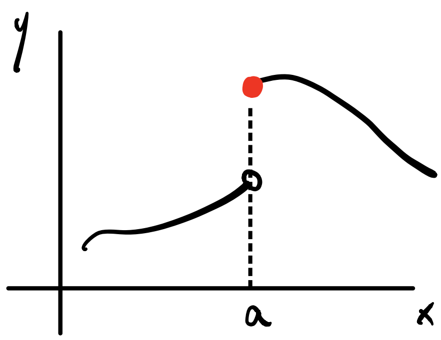
REMOVABLE DISCONTINUITY



JUMP

CONTINUOUS FROM THE LEFT

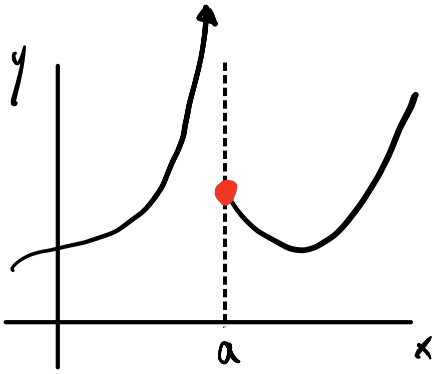
$$\lim_{x \to a^-} f(x) = f(a)$$



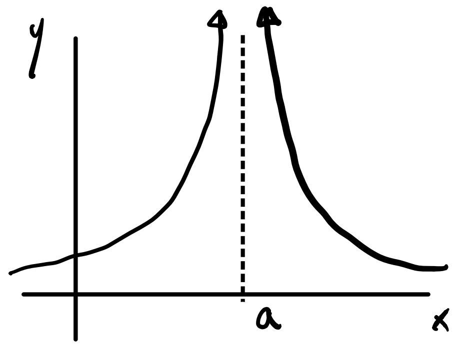
JUMP

CONTINUOUS FROM THE RIGHT

$$\lim_{x \to a^+} f(x) = f(a)$$



INFINITE

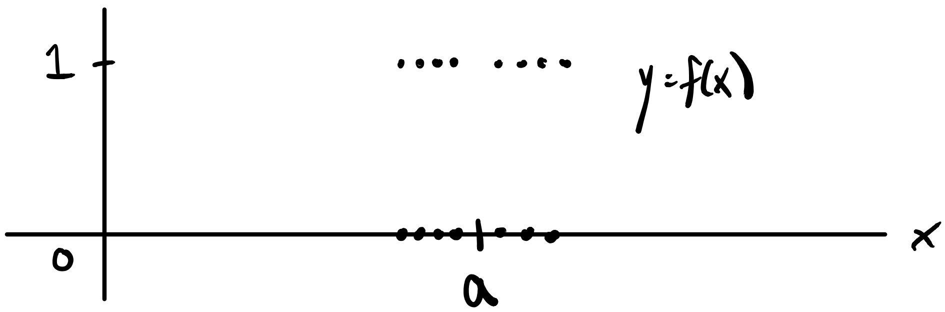


INFINITE

AT LEAST ONE ONE-SIDED LIMIT IS INFINITE.

CONSIDER $f(x) = \begin{cases} 1 & \text{IF } x \text{ IS RATIONAL } (x \in \mathbb{Q}) \\ 0 & \text{IF } x \text{ IS IRRATIONAL } (x \notin \mathbb{Q}) \end{cases}$

f IS DISCONTINUOUS AT EVERY POINT IN ITS DOMAIN.



PROPERTIES OF CONTINUOUS FUNCTIONS:

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

└ ALL OF THESE FOLLOW IMMEDIATELY FROM LIMIT LAWS:

e.g. $\lim_{x\to a} \frac{f(x)}{g(x)} \overset{LL3}{=} \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}$

ASSUMING $g(a) \neq 0$, THIS SHOWS $\frac{f}{g}$ IS CONTINUOUS AT a.

$\boxed{7}$ **Theorem** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

(RESTATEMENT OF DIRECT SUBSTITUTION RULE)

BASICALLY, EVERY ELEMENTARY FUNCTION YOU KNOW IS CONTINUOUS AT EVERY POINT IN ITS DOMAIN.

THE POINT: WHEN EVALUATING $\lim_{x\to a} f(x)$,

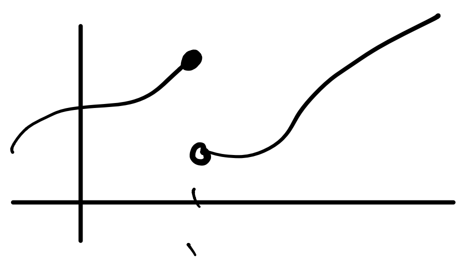
TRY SUBSTITUTING $x = a$.

IF $f(a)$ IS DEFINED $(a \in \mathrm{Dom}(f))$

THEN THAT'S THE ANSWER!

ex. **39–40** Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$



WE MUST SHOW THAT $\lim_{x \to a} f(x) = f(a)$

FOR ALL $-\infty < a < \infty$.

(1) IF $a < 1$: $\lim_{x \to a} f(x) = \lim_{x \to a} 1 - x^2 = 1 - a^2 = f(a)$ ✓

POLYNOMIALS ARE CONTINUOUS ON THEIR DOMAIN

(2) IF $a > 1$: $\lim_{x \to a} f(x) = \lim_{x \to a} \sqrt{x-1}$

$= \sqrt{\lim_{x \to a} x - 1} = \sqrt{a-1} = f(a)$ ✓

(3) IF $a = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 - x^2 = 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} \sqrt{x-1} = 0$

$\therefore \lim_{x \to 1} f(x) = 0 = f(1)$ ✓

DONE.

RECALL: $\lim_{x \to a} f(x) = L$ IF & ONLY IF

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

COMPOSITIONS OF CONTINUOUS FUNCTIONS:

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

AS $x \to a$, $g(x) \to b$

AND SINCE f IS CONTINUOUS AT b,

AS $g(x) \to b$, $f(g(x)) \to f(b)$

INPUT OF f

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

"COMPOSITION OF CONTINUOUS FUNCTIONS IS CONTINUOUS."

e.g. $f(x) = x^3 - 2x + 3$ (POLYNOMIAL) CONTINUOUS $\forall x \in \mathbb{R}$

$g(x) = \cos(x)$ (TRIG) CONTINUOUS $\forall x \in \mathbb{R}$

$\Rightarrow g \circ f(x) = g(f(x))$

$= \cos(x^3 - 2x + 3)$ CONTINUOUS $\forall x \in \mathbb{R}$

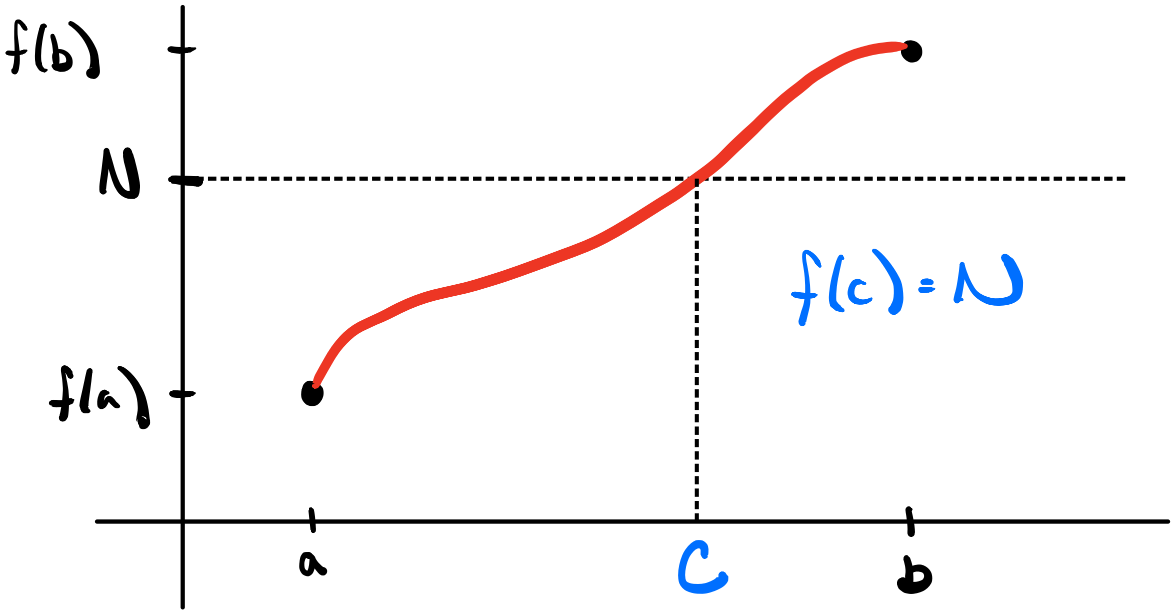
$f \circ g(x) = f(g(x)) = f(\cos(x))$

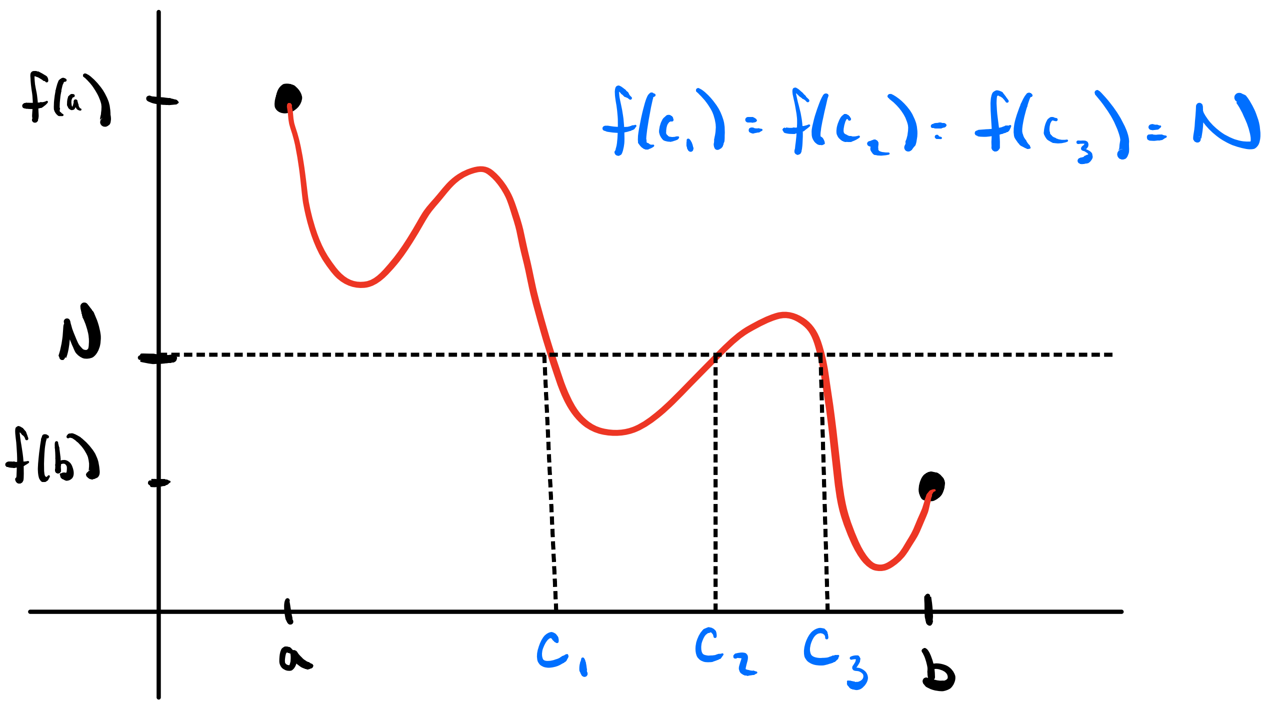
$= \cos^3(x) - 2\cos(x) + 3$

CONTINUOUS $\forall x \in \mathbb{R}$

The Intermediate Value Theorem

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.





ex.

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

53. $x^4 + x - 3 = 0$, $(1, 2)$

54. $2/x = x - \sqrt{x}$, $(2, 3)$

55. $\cos x = x$, $(0, 1)$

56. $\sin x = x^2 - x$, $(1, 2)$

57–58 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

57. $\cos x = x^3$ **58.** $x^5 - x^2 + 2x + 3 = 0$

#55. $\cos x = x$ SHOW $\exists$ SOL'N IN $(0,1)$

$\cos x - x = 0$

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

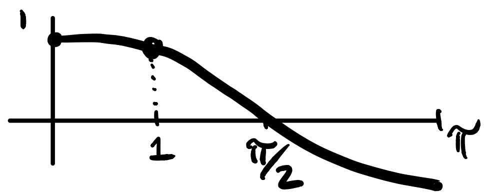
$f(x) = \cos x - x$

f IS CONTINUOUS ON $[0, 1]$

$f(0) = \cos(0) - 0 = 1$

$f(1) = \cos(1) - 1 =$

$< 1 - 1 = 0$



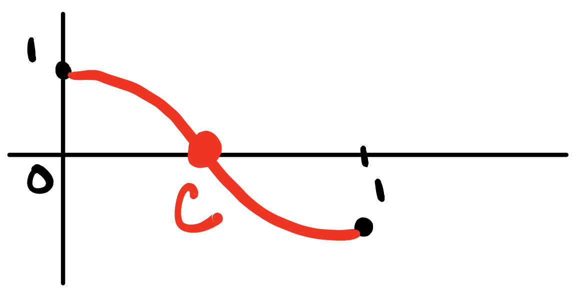
$f(1) < 0 < f(0)$

0 IS BETWEEN $f(0)$ & $f(1)$

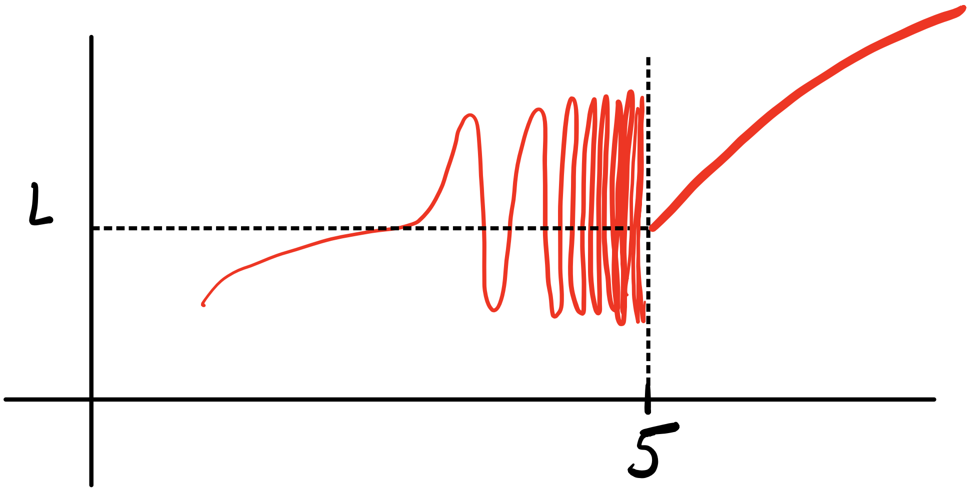
$\therefore \exists\ c \in (0,1)$ s.t. $f(c) = 0$

i.e. $\cos(c) - c = 0$

$\cos(c) = c$ ✓



$f(c) = 0$



$$\lim_{x \to 5^-} f(x) = L$$

FALSE