Solutions to MATH 1108 Final Exam
Fall 2021

(1) (10 points) Translate the following word problem into a mathematical linear programming problem. Do not solve it. *The Old-World Class Ring Company designs and sells two types of rings: brass and gold. They can produce up to 24 rings each day using up to 60 total person-hours of labor. It takes 3 person-hours to make one brass ring and 2 person-hours to make one gold ring. The profit on a brass ring is \$40 and on a gold ring is \$30. The company wants to know how many of each type of ring should be made daily to maximize their profit.*

Write a mathematical linear programming problem whose solution would tell the company how to accomplish their goal. You do not need to solve the problem you write.

Let x be the number of brass rings produced per day, and let y be the number of gold rings produced per day. The profit from the brass rings is $40x$, and the profit from the gold rings is $30y$. Thus, the total profit is $40x + 30y$. We seek to maximize profit, so $z = 40x + 30y$ is the objective function.

We have $x \geq 0$ and $y \geq 0$ since the company cannot produce negative numbers of rings. Since the company can produce up to 24 rings each day, we have $x + y \leq 24$. The labor required to make the brass rings is $3x$ hours, and the labor required to produce the gold rings is $2y$. Since there are most 60 person-hours of labor available, we have $3x + 2y \leq 60$. Thus the feasible region is given by the inequalities

$$\begin{aligned} x + y &\leq 24 \\ 3x + 2y &\leq 60 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(2) (10 points) For the following objective function and feasible region, determine whether the maximum and/or minimum exist, and find them if they exist.

Objective function: $z = 4x + 5y$
Feasible region: $x - y \leq 5$
$2x + y \geq 15$
$x \geq 0$
$y \geq 0$

First, note that the inequalities $x \geq 0$ and $y \geq 0$ tell us to work in the first quadrant.

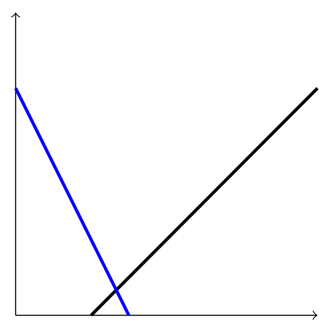
Next, rewrite the first two inequalities in slope-intercept form.

$$\begin{array}{lrcl} \text{Ineq. 1:} & x - y & \leq & 5 \\ & y & \geq & x - 5 \end{array}$$

$$\begin{array}{lrcl} \text{Ineq. 2:} & 2x + y & \geq & 15 \\ & y & \geq & -2x + 15 \end{array}$$

Next, we turn the inequalities into equations and graph the corresponding lines.

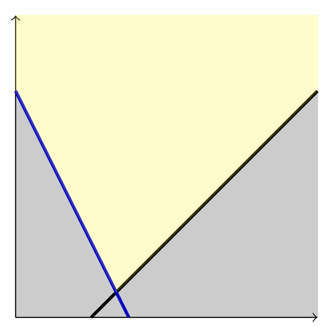
Inequality	Equation
$y \geq x - 5$	$L_1 : y = x - 5$
$y \geq -2x + 15$	$L_2 : y = -2x + 15$



Next, we highlight the halfspace satisfied by each inequality and cross out the halfspace that is not satisfied by the inequality.

- Since the inequality $y \geq x - 5$ is $\geq$ in slope-intercept form, we highlight the region above the black line, and cross out the region below.
- Since the inequality $y \leq -2x + 15$ is $\geq$ in slope-intercept form, we highlight the region above the blue line, and cross out the region below.

This yields the following picture:



We see that the feasible region is unbounded. It has three edges and 2 corner points.

From this, we deduce that objective function $z = 4x+5y$ has no maximum, because we can make x and y infinitely large.

On the other hand, the objective function has a minimum, and it must occur at one of the corner points.

One corner point is the y-intercept of the blue line, which is $(0, 15)$.

The other corner point is the intersection of the black and blue lines.

$$\begin{aligned}
L_1 : y &= x - 5 \\
L_2 : y &= -2x + 15 \\
x - 5 &= -2x + 15 \\
3x &= 20 \\
x &= \frac{20}{3} \\
y &= \frac{20}{3} - 5 = \frac{5}{3}
\end{aligned}$$

This corner point is $(\frac{20}{3}, \frac{5}{3})$.

Lastly, we make a table and compute the value of the objective function on each corner point.

(x, y)	$z = 4x + 5y$
$(0, 15)$	75
$(\frac{20}{3}, \frac{5}{3})$	$\frac{105}{3}$

The minimum value is $z = \frac{105}{3}$, which occurs at the corner point $(\frac{20}{3}, \frac{5}{3})$.

(3) (5 points) How long will it take \$20,000 to grow to \$31,000 if it is invested at 5% compounded quarterly?

Since the problem specifies that interest is compounding, and there is only one payment (not an annuity), we use the formula $A = P(1 + i)^n$. We have $A = 31,000$, $P = 20,000$, $r = 0.05$, $m = 4$, $i = r/m = 0.0125$, and n is the unknown.

$$\begin{aligned}
A &= P(1 + i)^n \\
31000 &= 20000(1 + 0.0125)^n \\
20000(1 + 0.0125)^n &= 31000 \\
(1.0125)^n &= \frac{31}{20} \\
\ln((1.0125)^n) &= \ln\left(\frac{31}{20}\right) \\
n \ln(1.0125) &= \ln\left(\frac{31}{20}\right) \\
n &= \frac{\ln\left(\frac{31}{20}\right)}{\ln(1.0125)}
\end{aligned}$$

Finally, after computing this quantity, we would round up to find the number of quarters required. (Using a calculator, we get $n \approx 35.27$, which rounds up to 36 quarters, or 9 years.)

(4) (5 points) Ian deposits \$100 in a savings account at the end of each month for 7 years, where the interest rate is 3% compounded monthly. How much interest does he earn over this period?

This description involves multiple equally spaced payments, that is, an annuity, and we want to know the interest earned. For this, we need to know the future value of the annuity.

We have PMT $= 100$, $r = 0.03$, $m = 12$, so $i = r/m = 0.03/12 = 0.0025$, and $n = 7 \times 12 = 84$. We compute

$$\mathrm{FV}_{\text{ann}} = \mathrm{PMT}\frac{((1+i)^n - 1)}{i}$$
$$\mathrm{FV}_{\text{ann}} = 100\frac{((1+0.0025)^{84} - 1)}{0.0025}$$

Ian makes 84 payments of \$100 each, for a total of \$8400 in payments. The difference between the future value and his payments is the amount of interest earned:

$$\text{interest} = 100\frac{((1+0.0025)^{84} - 1)}{0.0025} - 8400$$

(Using a calculator, we get $\mathrm{FV}_{\text{ann}} \approx \$9334.19$, and therefore the interest earned is \$934.19.)

(5) (10 points) Jack and Jill take out a 25-year loan for \$400,000 to buy a house. The annual rate is 4.6% compounded semiannually. Let PMT represent their semiannual payment.
   (a) Write a calculator-ready expression for PMT.
   (b) If they sell their house after 10 years, what is the unpaid balance of the loan? You may write your answer in terms of the variable PMT.

This description involves multiple equally spaced payments, that is, an annuity, and we know the value of the house at the time it is purchased, so we choose the formula for the present value of an annuity.

Here $\mathrm{PV}_{\text{ann}} = 400000$, PMT is the unknown, $r = 0.046$, $m = 2$, $i = r/m = 0.023$, and $n = 25 \times 2 = 50$. We compute

$$\mathrm{PV}_{\text{ann}} = \mathrm{PMT}\frac{(1-(1+i)^{-n})}{i}$$
$$400000 = \mathrm{PMT}\frac{(1-(1+0.023)^{-50})}{0.023}$$
$$\mathrm{PMT} = \frac{400000 \times 0.023}{(1-(1+0.023)^{-50})}$$

(Using a calculator, we get PMT $\approx \$13545.11$.)

To compute the unpaid balance, we put the number of payments remaining into the present value for annuities formula. When they are 10 years into a 25 year loan, there are 15 years remaining. Since the payments are made semiannually, this means there are 30 payments remaining.

$$\text{unpaid balance} = \mathrm{PMT}\frac{(1-(1+i)^{-n})}{i}$$
$$\text{unpaid balance} = \mathrm{PMT}\frac{(1-(1+0.023)^{-30})}{0.023}$$

(Using a calculator, we get that the unpaid balance is approximately \$291,213.13.)

(6) (5 points) A 10-person team is interviewing candidates for an internship. The team includes 4 directors and 6 analysts. The interview committee will be composed of 2 directors and 2 analysts. How many different committees are possible?

There are ${}_4C_2$ ways to choose 2 directors, and ${}_6C_2$ ways to choose 2 analysts. Therefore, by the Multiplication Principle, there are ${}_4C_2 \times {}_6C_2$ possible committees.

(7) (5 points) 10 teams of students need to present their final projects in a class. The presentations will take place over two class periods, with five presentations each period. How many different schedules are possible for the first period of presentations?

Here, the order of the presentations matters, so we are dealing with permutations. The number of permutations of 5 teams out of 10 possible is ${}_{10}P_5$.

(8) (5 points) Find the probability of getting a hand of 2 Kings, 2 Queens, and 1 Jack (not necessarily in that order) when you draw 5 cards from a standard 52-card deck without replacement.

The sample space is five card hands from a standard deck. The number of such hands is ${}_{52}C_5$.

There are four kings in the deck, so the number of ways to draw 2 kings is ${}_4C_2$. There are four queens in the deck, so the number of ways to draw 2 queens is ${}_4C_2$. There are four jacks in the deck, so the number of ways to draw 1 jack is ${}_4C_1 = 4$. By the Multiplication Principle, there are ${}_4C_2 \times {}_4C_2 \times {}_4C_1$ hands that contain 2 kings, 2 queens, and 1 jack.

Therefore, the probability of getting such a hand is

$$P(E) = \frac{n(E)}{n(S)} = \frac{{}_4C_2 \times {}_4C_2 \times {}_4C_1}{{}_{52}C_5}$$

(9) (5 points) A watchmaker offers its customers a 10-year warranty on new watches. Their data indicates that in this time, 5% of customers will require a full replacement, an additional 15% of customers will require one battery replacement, and a further 5% of customers will require two battery replacements. A full replacement costs $200, and each battery replacement costs $15. What is the expected cost per watch of this warranty program?

First, we can write the distribution of the service costs.

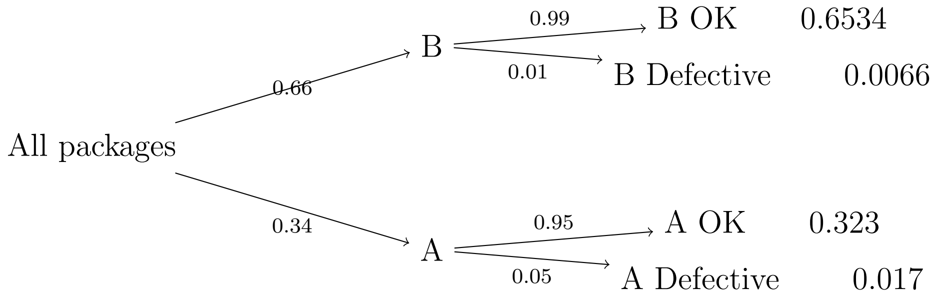
Service	Cost	Probability
Full replacement	200	0.05
One battery replacement	15	0.15
Two battery replacements	30	0.05
None	0	0.75

The expected cost per watch is

$$200 \times 0.05 + 15 \times 0.15 + 30 \times 0.05 + 0 \times 0.75$$

(10) (5 points) Two stores sell a certain model of MP3 player. Store A has 34% of the sales, 5% of which are of defective items, and Store B has 66% of the sales, 1% of which are of defective items. The difference in defective rates is due to different levels of pre-sale checking of the product. A person receives a defective item of this product as a gift. What is the probability it came from Store B?

We can either draw a tree or use Bayes's Theorem. Using a tree diagram:



$$P(\text{B}|\text{Defective}) = \frac{P(\text{B Defective})}{P(\text{Defective})} = \frac{0.0066}{0.017 + 0.0066} \approx 0.2797$$

(11) (10 points) A botanist wants to grow a rare plant in his greenhouse. The probability that a given bulb will mature is 0.42. Suppose 6 bulbs are planted. Assuming the bulbs are independent of one another, what is the probability that

(a) Exactly 4 bulbs will mature
(b) 3 or more bulbs will mature

For each bulb, there are two outcomes: either it matures, or it does not. We can consider each bulb as a Bernoulli trial, and consider the bulb maturing as a success. The probability of success in each trial is $p = 0.42$, and the probability of failure is $q = 0.58$. There are 6 trials, and the trials are independent of each other. The probability of x successes in n trials is

$$P(x \text{ successes}) = {}_nC_x \cdot p^x \cdot q^{n-x}$$

Part (a) asks us for the probability of four successes in six trials. With $n = 6$, $x = 4$, $p = 0.42$, and $q = 0.58$, this is

$$P(3 \text{ successes}) = {}_6C_4 \cdot 0.42^4 \cdot 0.58^2$$

(Using a calculator, this is approximately 0.1570.)

In part (b), we need to consider x for all values greater than or equal to 3. We have

$$\begin{aligned} P(3 \text{ successes}) &= {}_6C_3 \cdot 0.42^3 \cdot 0.58^3 \\ P(4 \text{ successes}) &= {}_6C_4 \cdot 0.42^4 \cdot 0.58^2 \\ P(5 \text{ successes}) &= {}_6C_5 \cdot 0.42^5 \cdot 0.58^1 \\ P(6 \text{ successes}) &= {}_6C_6 \cdot 0.42^6 \cdot 0.58^0 \end{aligned}$$

So the probability that three or more bulbs mature is

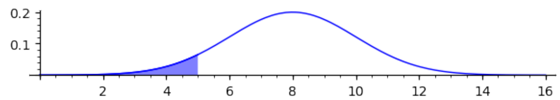
$${}_6C_3 \cdot 0.42^3 \cdot 0.58^3 + {}_6C_4 \cdot 0.42^4 \cdot 0.58^2 + {}_6C_5 \cdot 0.42^5 \cdot 0.58^1 + {}_6C_6 \cdot 0.42^6 \cdot 0.58^0$$

(Using a calculator, this is approximately 0.4971.)

(12) (10 points) The mean clotting time of blood is 8 seconds, with a standard deviation of 2 seconds. Suppose that the clotting time of blood is approximated by a normal curve. What is the probability that blood clotting time will be

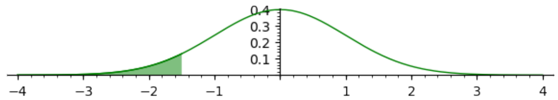
(a) Less than 5 seconds
(b) At least 10.5 seconds

Part (a): First, we sketch the region described.

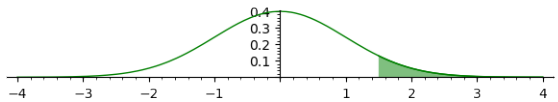


It is a left tail.

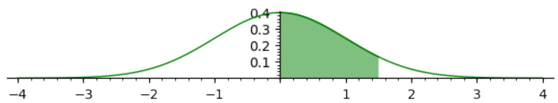
Next we compute the z-score of 5. We get $z = \frac{x-\mu}{\sigma} = \frac{5-8}{2} = -1.5$. Thus, the area of the left tail shown on the normal curve above is equal to the area of the left tail shown on the standard normal curve below.



By symmetry, this left tail has the same area as a right tail beginning at z-score +1.5:



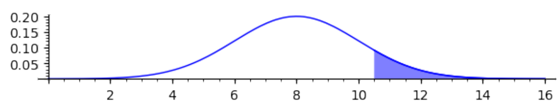
We can look up the area of the strip from z=0 to z=1.5 in the table from Appendix C:



By looking in row 1.5, column .00, we find that this area is 0.4332.

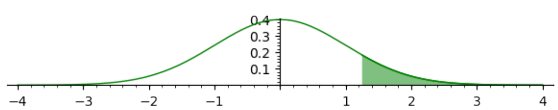
Finally, we can subtract this from 0.5 (the area of the right half of the normal curve) to get the area of the right tail: $0.5 - 0.4332 = 0.0668$. In summary: the probability that the blood clots in 5 seconds or less is 0.0668.

Part (b): First, we sketch the region described.

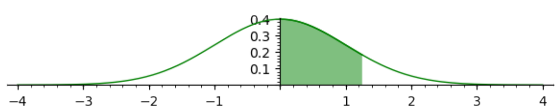


It is a right tail.

Next we compute the z-score of 10.5. We get $z = \frac{x-\mu}{\sigma} = \frac{10.5-8}{2} = 1.25$. Thus, the area of the right tail shown on the normal curve above is equal to the area of the right tail shown on the standard normal curve below.



We can look up the area of the strip from z=0 to z=1.25 in the table from Appendix C:



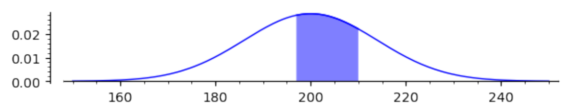
By looking in row 1.2, column .05, we find that this area is 0.3994.

Finally, we can subtract this from 0.5 (the area of the right half of the normal curve) to get the area of the right tail. $0.5 - 0.3994 = 0.1006$. In summary: the probability that the blood clots in 10.5 seconds or more is 0.1006.

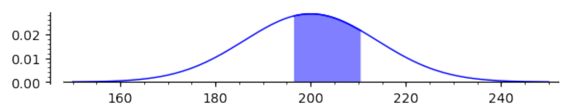
(13) (10 points) 2% of flat irons produced at a certain plant are defective. Estimate the probability that of 10,000 randomly selected flat irons, the number of defective irons is between 197 and 210 inclusive.

For each flat iron, there are two outcomes: either it is defective, or not. We can therefore view each flat iron as a Bernoulli trial. If we label success as finding a defective iron, then the probability of success is $p = 0.02$ and the probability of failure is $q = 0.98$. Thus, the number of defective irons out of 10000 irons is a binomial random variable. Since n is very large and p is not too small, we may approximate this binomial random variable by a normal random variable with mean $\mu = np$ and standard deviation $\sigma = \sqrt{npq}$. We have $\mu = 10000 \times 0.02 = 200$ and $\sigma = \sqrt{10000 \times 0.02 \times 0.98} = \sqrt{196} = 14$.

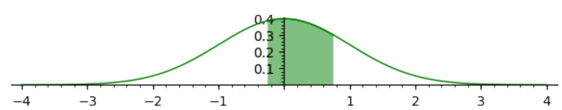
The problem asks for the probability that between 197 and 210 flatirons are defective. First, we sketch this region:



Since we are approximating a binomial random variable by a normal random variable, we increase the region by 0.5 units in both directions. Thus, the new region extends from 196.5 to 210.5:

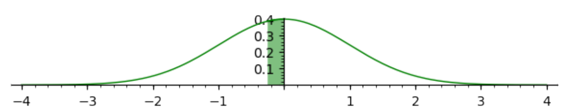


Next, we compute the z-scores of these boundaries. The z-score of 196.5 is $z = \frac{x-\mu}{\sigma} = \frac{196.5-200}{14} = -\frac{3.5}{14} = -0.25$. The z-score of 210.5 is $z = \frac{x-\mu}{\sigma} = \frac{210.5-200}{14} = \frac{10.5}{14} = 0.75$. Thus, we seek the area of the strip under the standard normal curve ranging from $z = -0.25$ to $z = 0.75$:

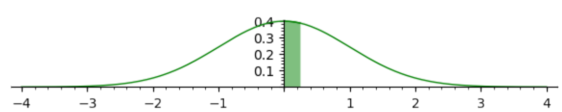


To get this area, we split it into two strips: a strip from $z = 0$ to $z = -0.25$, and a strip from $z = 0$ to $z = 0.75$.

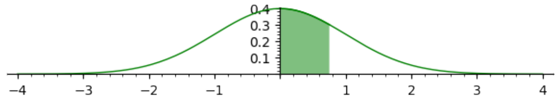
By symmetry, the strip from $z = -0.25$ to $z = 0$ has the same area as the strip from $z = 0$ to $z = +0.25$



This area can be found by looking up $z = +0.25$ in the table from Appendix C. We look in row 0.2 and column .05 to find 0.0987.



To get the area of the strip from $z = 0$ to $z = 0.75$, we look in row 0.7 and column .05, and find 0.2734.



Putting these together, we find that the area of the region from $z = -0.25$ to $z = 0.75$ is $0.0987 + 0.2734 = 0.3721$. In summary, the probability that between 197 and 210 defective flatirons are found is 0.3721.