1 Bernoulli Trials and Binomial Experiments

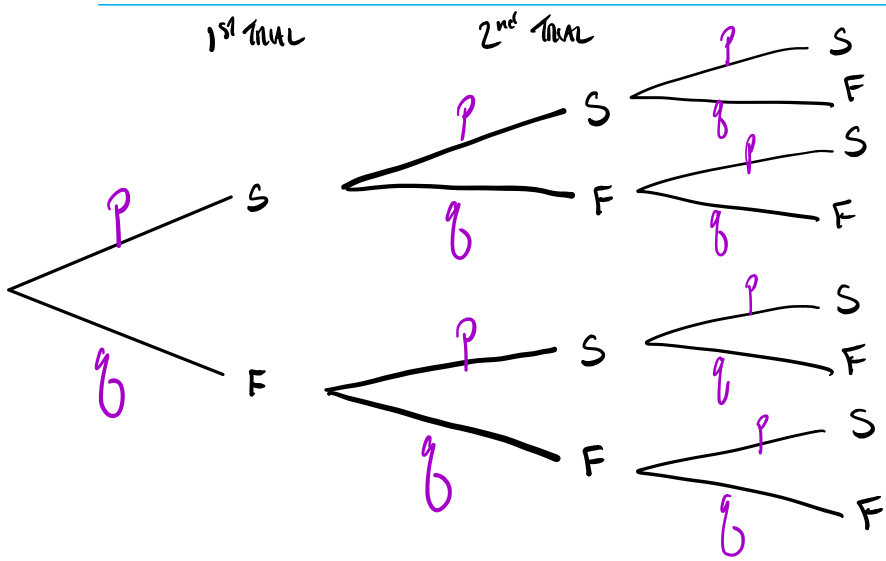
DEFINITION Bernoulli Trials

A sequence of experiments is called a **sequence of Bernoulli trials**, or a **binomial experiment**, if

1. Only two outcomes are possible in each trial.
2. The probability of success p for each trial is a constant (probability of failure is then $q = 1 - p$).
3. All trials are independent.

Definition A **binomial experiment** is one that has these five characteristics:

1. The experiment consists of n identical trials. BERNOULLI TRIALS
2. Each trial results in one of two outcomes. For lack of a better name, the one outcome is called a success, S, and the other a failure, F.
3. The probability of success on a single trial is equal to p and remains the same from trial to trial. The probability of failure is equal to $(1 - p) = q$.
4. The trials are independent.
5. We are interested in x, the number of successes observed during the n trials, for $x = 0, 1, 2, \ldots, n$.

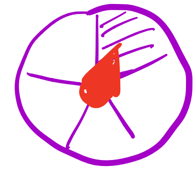


IMPORTANT #'s FOR ANY BINOMIAL EXPERIMENT:

n # TRIALS

p SUCCESS

q FAILURE



1. Label each of the following experiments as binomial or not binomial.
   - (a) A single coin is flipped repeatedly until a head is observed and x is the number of flips.
   - (b) Seven cards are dealt from a shuffled deck of 52 cards and x is the number of aces dealt.
   - (c) Due to a pandemic, only 1 out of every 5 customers is allowed into a particular store. Sarah visit this store on 7 consecutive days and x is the number times she is allowed into the store. SUCCESS (COUNT)
   - (d) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar simulteously and x is the number of red marbles.
   - (e) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar, replacing the marble after each selection, and x is the number of red marbles.

$n = 7$
$p = .2$
$q = .8$

$n = 20$
$p = 12/20$
$q = 8/20$

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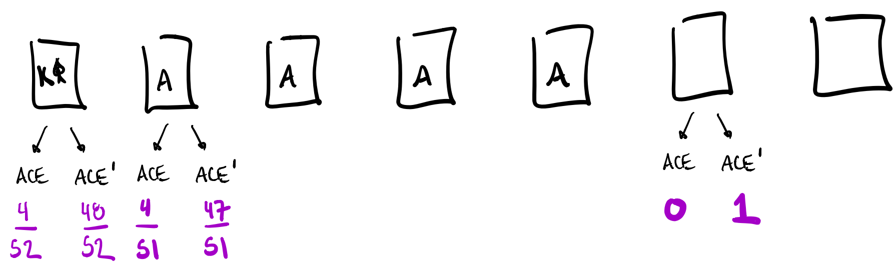
MIN VALUE

MAX VALUE

IMP. #'s n, p, q

FINITE # POSSIBLE OUTCOMES.

(b)

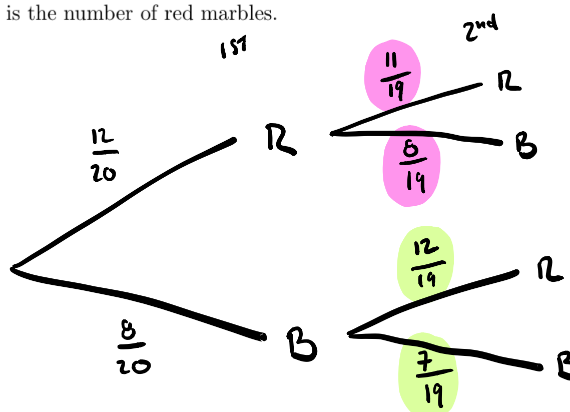


$n = 7$
p
q

TRIALS ARE NOT INDEPENDENT.

(d) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar simulteously and x is the number of red marbles.

(d)



TRIALS ARE NOT INDEPENDENT!

DEFINITION Binomial Distribution

$$P(X_n = x) = P(x \text{ successes in } n \text{ trials})$$
$$= {}_nC_x p^x q^{n-x} \qquad x \in \{0, 1, 2, \ldots, n\}$$

where p is the probability of success and q is the probability of failure on each trial. Informally, we will write $P(x)$ in place of $P(X_n = x)$.

2. Imagine two different six-sided fair dice, called die A and die B.

- Die A has its faces labeled 1, 1, 1, 2, 2, 3.
- Die B has its faces labeled 1, 2, 2, 3, 3, 3.

SUCCESS

SUCCESS

(a) What is the probability that die A is rolled 5 times and a 2 appears exactly 3 times?

You TRY!

(b) What is the probability that die B is rolled 12 times and a 1 appears exactly 3 times?

(a) $n = 5$

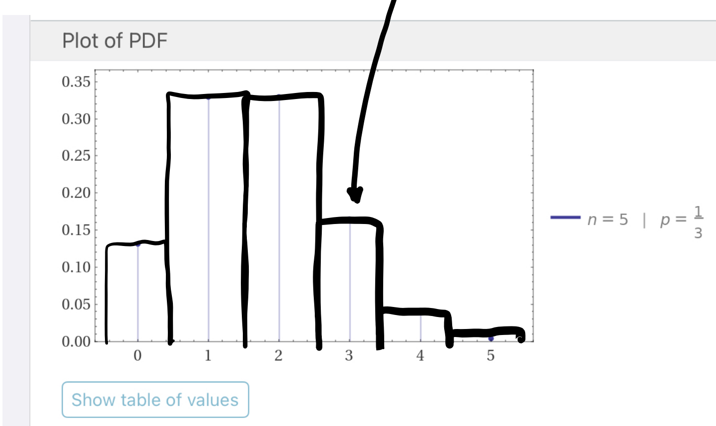
$p = \frac{2}{6} = \frac{1}{3}$

$q = \frac{4}{6} = \frac{2}{3}$

$$P(X = k) = {}_nC_k \, p^k q^{n-k}$$

$$P(X = 3) = {}_5C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{\overbrace{5-3}^{2}} = .1646$$

$0 \leq \text{ANY \#} \leq 5$



(b) $n = 12$

$p = 1/6$

$q = 5/6$

#SUCCESSES #FAILURES

$$P(x = 3) = {}_{12}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^9$$

$= .1974$

3. Imagine two different six-sided fair dice, called die A and die B.

- Die A has its faces labeled 1, 1, 1, 2, 2, 3.
- Die B has its faces labeled 1, 2, 2, 3, 3, 3.

SUCCESS

(a) What is the probability that die A is rolled 4 times and a 1 appears exactly 4 times, given that a 1 appears at least 3 times?

CHALLENGE!

(b) What is the probability that die B is rolled 6 times and the numbers 1, 2, and 3 each appear exactly twice?

(a) $n = 4$

$p = \frac{1}{2}$

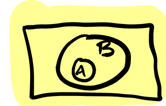
$q = \frac{1}{2}$

LET A = 1 APPEARS 4 TIMES

B = 1 APPEARS ≥ 3 TIMES

$A \cap B = A$

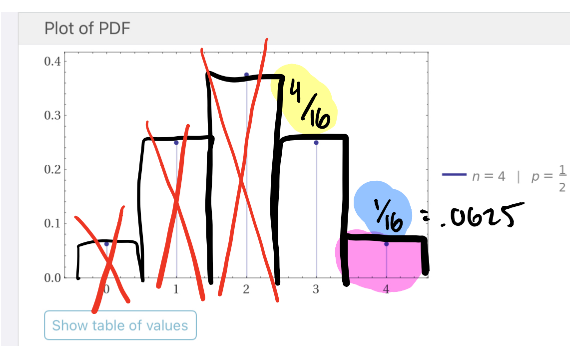
$A \subseteq B$



FIND $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/16}{5/16}$$

CONDITIONAL PROBABILITY

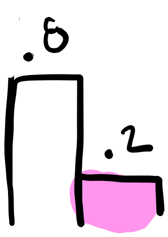
$= .2$



$$P(A \cap B) = P(A) = {}_4C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^0$$

$$= \frac{1}{16}$$

UPDATED SAMPLE SPACE



$$P(B) = P(x = 3 \cup x = 4)$$

$$= P(x = 3) + P(x = 4)$$

$$= {}_4C_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^1 + \frac{1}{16} = \frac{4}{16} + \frac{1}{16} = \frac{5}{16}$$

When x is the number of successes in a series of n Bernoulli trials, the mean and standard deviation for x are

$$\mu = np, \qquad \sigma = \sqrt{npq}.$$

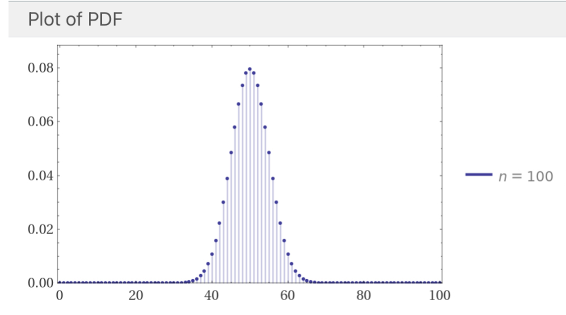
4. Let x represent be the number of success in 20 Bernoulli trials, each with probability of success $p = .85$. Find the mean (i.e. expected value) and standard deviation for x.

ON AVERAGE, # SUCCESSES IN 20 TRIALS IS 85% OF 20

$$\mu = np = (20)(.85) = 17$$

$$\sigma = \sqrt{npq} = \sqrt{(20)(.85)(.15)} = \sqrt{2.55} \approx 1.5969$$

BINOMIAL, $n = 100$ $p = .5$ ~ NORMAL



2 Normal Distributions

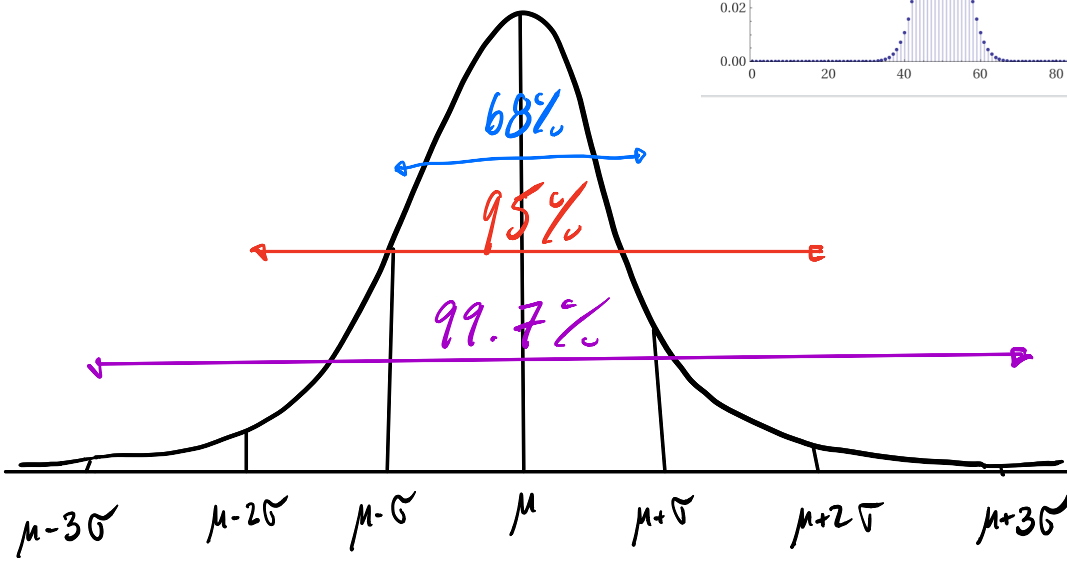
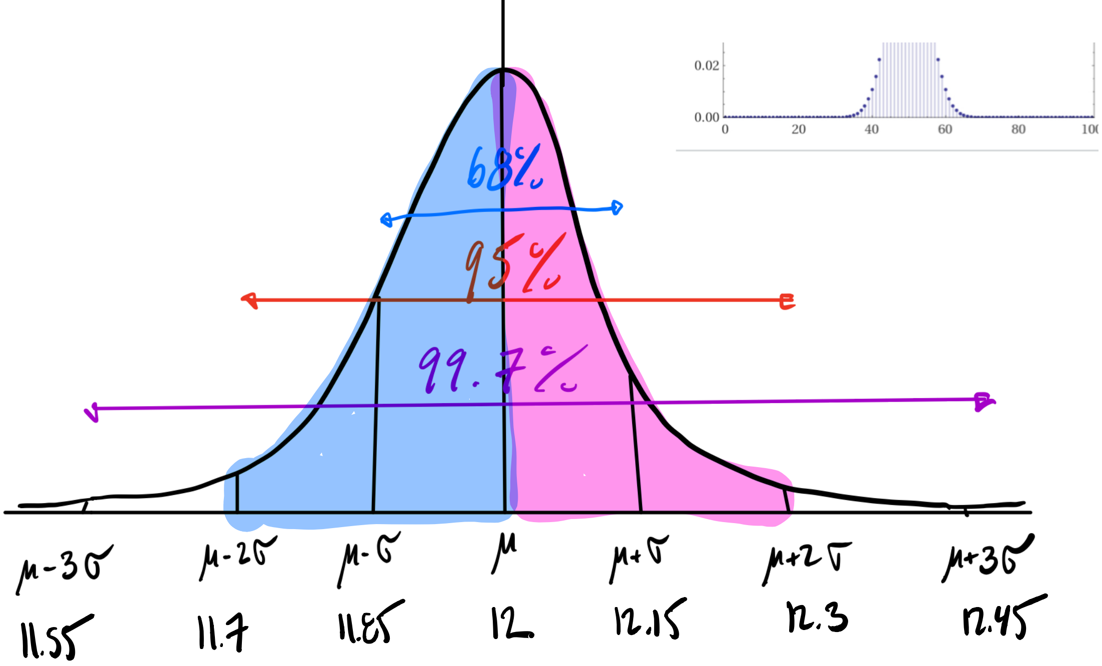


Figure 1: The 68-95-99.7 rule for normal distributions.

5. A machine in a bottling plant is set to dispense 12 oz of soda into cans. The machine is not perfect, and so every time the machine dispenses soda, the exact amount dispensed is a number x with a normal distriution. The mean and standard deviation for x are $\mu = 12$ oz and $\sigma = 0.15$ oz, respectively. Approximate the following probabilities using the 68-95-99.7 rule.
   (a) Give a range of values such that the amount of soda in 68% of all cans filled by this machine are in this range.
   (b) Give a range of values such that the amount of soda in 95% of all cans filled by this machine are in this range.
   (c) Give a range of values such that the amount of soda in 99.7% of all cans filled by this machine are in this range.
   (d) $P(11.85 \le x \le 12.15)$
   (e) $P(11.70 \le x \le 12)$
   (f) $P(x \le 11.70)$
   (g) $P(12.3 \le x \le 12.45)$
   (h) $P(x \le 12 \cup x \ge 12.45)$

SYMMETRIC



(a) Give a range of values such that the amount of soda in 68% of all cans filled by this machine are in this range. [11.85 , 12.15]

(b) Give a range of values such that the amount of soda in 95% of all cans filled by this machine are in this range. [11.7 , 12.3]

(c) Give a range of values such that the amount of soda in 99.7% of all cans filled by this machine are in this range.

~~(d)~~ $P(11.85 \leq x \leq 12.15)$

(e) $P(11.70 \leq x \leq 12)$ .475 ← HALF OF $P(11.7 \leq x \leq 12.3)$

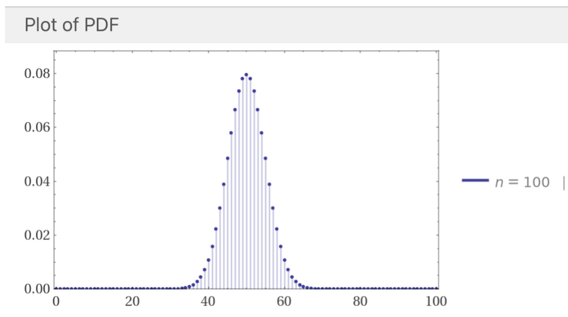
(f) $P(x \leq 11.70)$

(g) $P(12.3 \leq x \leq 12.45)$

(h) $P(x \leq 12 \cup x \geq 12.45)$

Sometimes binomial distributions have the same shape as normal distriutions.

6. An experiment is composed of flipping a fair coin 100 times and counting the number of heads that appear x. Use a normal distribution and the 68-95-99.7 rule to provide rough estimates for the probabilities of the following events.
   (a) You observe between 45 and 55 heads.
   (b) You observe more than 60 heads.



WE'LL DISCUSS THIS IN MODULE 10.