1 Bernoulli Trials and Binomial Experiments

DEFINITION Bernoulli Trials

A sequence of experiments is called a **sequence of Bernoulli trials**, or a **binomial experiment**, if

1. Only two outcomes are possible in each trial.
2. The probability of success p for each trial is a constant (probability of failure is then $q = 1 - p$).
3. All trials are independent. (IDENTICAL : SAME p & q)

FLIP COIN, ROLL DICE 1 NOT 1

BUILT OUT OF n BERNOULLI TRIALS

Definition A **binomial experiment** is one that has these five characteristics:

1. The experiment consists of n identical trials.
2. Each trial results in one of two outcomes. For lack of a better name, the one outcome is called a success, S, and the other a failure, F.
3. The probability of success on a single trial is equal to p and remains the same from trial to trial. The probability of failure is equal to $(1 - p) = q$.
4. The trials are independent.
5. We are interested in x, the number of successes observed during the n trials, for $x = 0, 1, 2, \ldots, n$.

POSSIBLE VALUES FOR X : $0, 1, 2, 3, \ldots, n$.
MIN MAX

1. Label each of the following experiments as binomial or not binomial.

   (a) A single coin is flipped repeatedly until a head is observed and x is the number of flips. No. n not def.

   (b) Seven cards are dealt from a shuffled deck of 52 cards and x is the number of aces dealt. No. Not ind.

   (c) Due to a pandemic, only 1 out of every 5 customers is allowed into a particular store. Sarah visit this store on 7 consecutive days and x is the number times she is allowed into the store.

   (d) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar simulteously and x is the number of red marbles.

   (e) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar, replacing the marble after each selection, and x is the number of red marbles.

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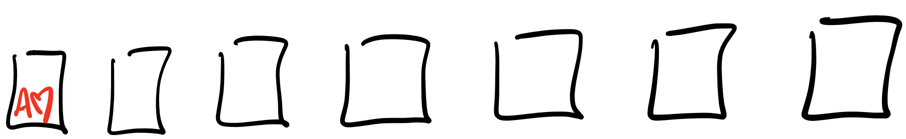
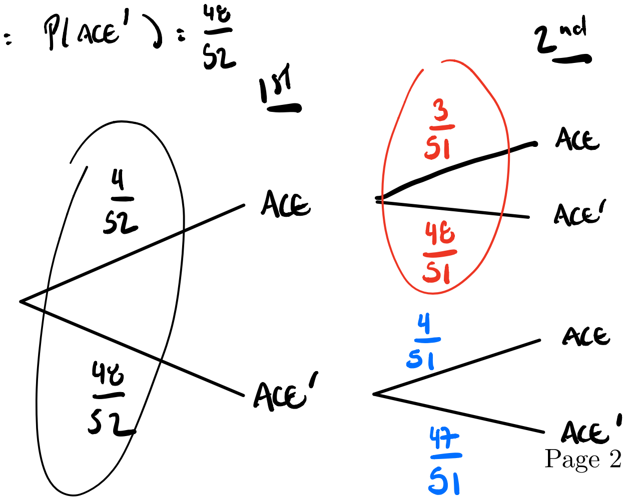
n must be defined!
(established in advance)

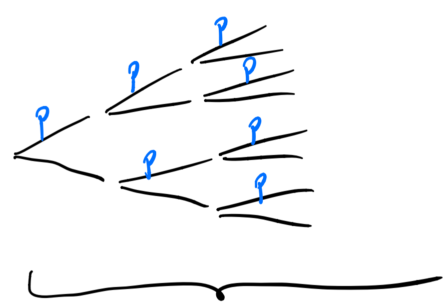
(b) $n = 7$



p: $p(\text{Ace}) = \frac{4}{52}$

q: $P(\text{Ace}') = \frac{48}{52}$





Binomial experiments:
identical, indep trials
(same p & q every time)

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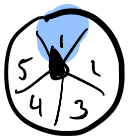
✗ (b) Seven cards are dealt from a shuffled deck of 52 cards and x is the number of aces dealt.



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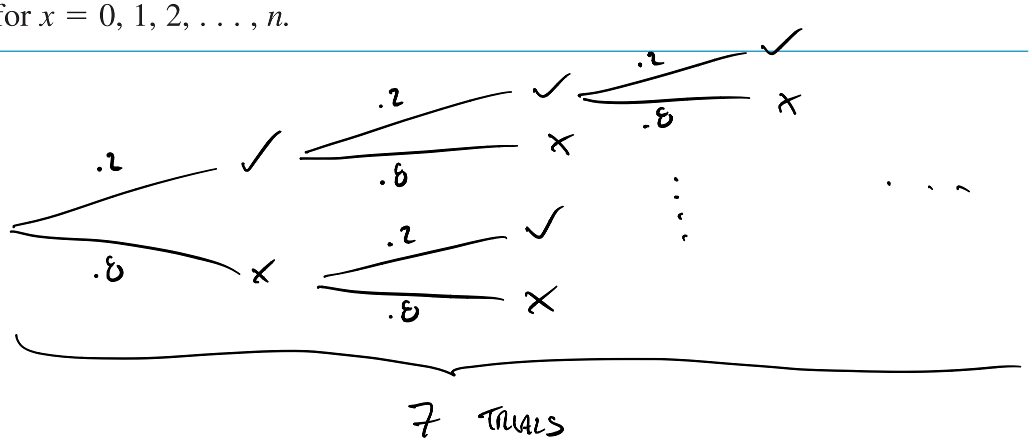
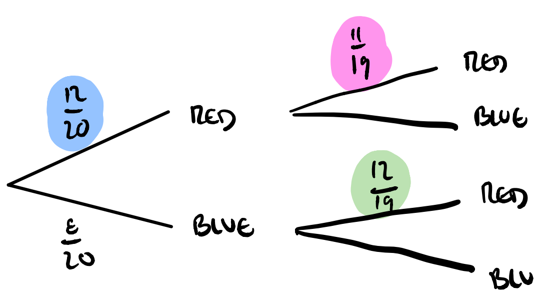


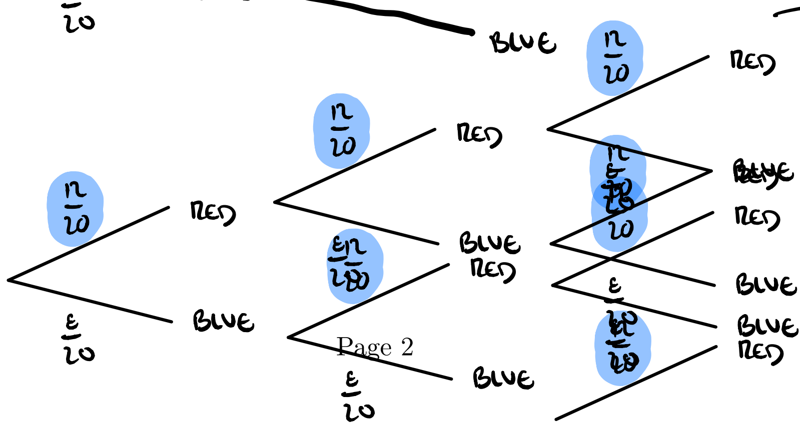
(d) WITHOUT REPLACEMENT



p = CHANGING EACH MARBLE

NOT BINOMIAL

(e) WITH REPLACEMENT



DEFINITION **Binomial Distribution**

$$P(X_n = x) = P(x \text{ successes in } n \text{ trials})$$
$$= {}_nC_x p^x q^{n-x} \qquad x \in \{0, 1, 2, \ldots, n\}$$

where p is the probability of success and q is the probability of failure on each trial.
Informally, we will write $P(x)$ in place of $P(X_n = x)$.

2. Imagine two different six-sided fair dice, called die A and die B.
   - Die A has its faces labeled 1, 1, 1, 2, 2, 3.
   - Die B has its faces labeled 1, 2, 2, 3, 3, 3.

   Which of the following events is more likely? Why?

   (a) Roll die A 5 times and roll a 2 exactly 3 times.

   (b) Roll die B 12 times and roll a 3 exactly 7 times.

   (c) Roll both dice simulataneously 9 times and roll doubles exactly 6 times.

EXP. $n = 3$ TRIALS (EACH SUCCESS S, FAILURE F)

SAMPLE SPACE : {
SSS, ← 1 WAY TO GET 3 SUCC
FSS, SFS, SSF, ← 3 WAYS TO GET 2 SUCC.
SFF, FSF, FFS, ← 3 WAYS TO GET 1 SUCC.
FFF } ← 1 WAY TO GET 0 SUCC

$P(FSS) = qpp = p^2q$

$P(2 \text{ successes}) = P(FSS) + P(SFS) + P(SSF)$

$= p^2q + p^2q + p^2q = 3p^2q$

DISTINCT ARRANGEMENTS OF k SUCCESSES & n-k FAILURES.

$$P(x=k) = \underline{{}_nC_k}\, p^k q^{n-k}$$

WAYS TO GET k SUCCESSES IN n TRIALS

$$\underbrace{\underline{p}\;\underline{\quad}\;\underline{p}\;\underline{\quad}\;\underline{p}\;\underline{\quad}\;\underline{p}}$$

$n = 7$ TRIALS

$k = 4$ SUCCESSES

$${}_7C_4$$

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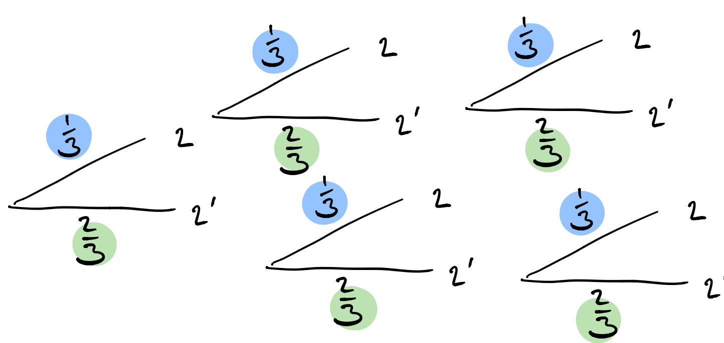
(a) BINOMIAL EXP. $n = 5$ TRIALS, COUNT 2's $\Rightarrow$ SUCCESS = ROLL 2

$P(\text{SUCCESS}) = p = \frac{2}{6} = \frac{1}{3}$

$P(\text{FAILURE}) = q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$



. . . IDENTICAL TRIALS.

$$P(X = k) = {}_nC_k \, p^k q^{n-k}$$

$$P(X = 3) = {}_5C_3 \left(\tfrac{1}{3}\right)^3 \left(\tfrac{2}{3}\right)^2 = .1646$$

DEFINITION Binomial Distribution

$$P(X_n = x) = P(x \text{ successes in } n \text{ trials})$$
$$= {}_nC_x p^x q^{n-x} \qquad x \in \{0, 1, 2, \ldots, n\}$$

where p is the probability of success and q is the probability of failure on each trial. Informally, we will write $P(x)$ in place of $P(X_n = x)$.

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(b) $n = 12$, $p = \frac{3}{6} = \frac{1}{2}$, $q = 1 - p = \frac{1}{2}$

$$P(x=7) = {}_nC_x\, p^x q^{n-x} = {}_{12}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^{12-7} = {}_{12}C_7 \left(\tfrac{1}{2}\right)^{12}$$

$$= .1934$$

(c) $n = 9$, $p = \frac{5}{18}$ $\qquad q = 1 - \frac{5}{18} = \frac{13}{18}$

$$P(x=6) = {}_9C_6 \left(\tfrac{5}{18}\right)^6 \left(\tfrac{13}{18}\right)^3$$

DOUBLES: $\overset{A}{(1}, \overset{B}{1)}$ OR $(\overset{A}{2}, \overset{B}{2})$ OR $(\overset{A}{3}, \overset{B}{3})$

$$\left(\tfrac{1}{2}\right)\left(\tfrac{1}{6}\right) + \left(\tfrac{1}{3}, \tfrac{1}{3}\right) + \left(\tfrac{1}{6}\right)\left(\tfrac{1}{2}\right)$$

$$= \frac{5}{18}$$

When x is the number of successes in a series of n Bernoulli trials, the mean and standard deviation for x are as follows.

ex. WE ALL FLIP A COIN 10 TIMES.
COUNT x = # HEADS.
SHARE RESULTS.

Mean: $\mu = np$

Standard deviation: $\sigma = \sqrt{npq}$

3. Let x represent be the number of success in 20 Bernoulli trials, each with probability of success $p = .85$. Find the mean (i.e. expected value) and standard deviation for x.

MEAN $\mu = (.85)(20) = 17$

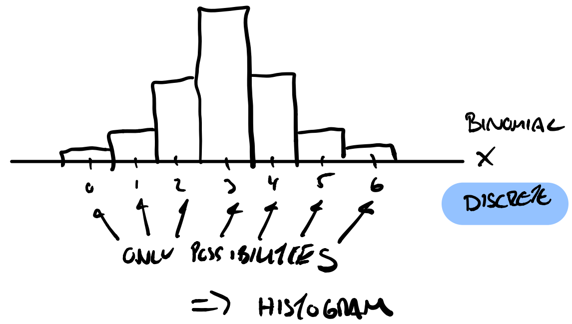
"EXPECTED VALUE"

STANDARD DEVIATION $\sigma = \sqrt{(20)(.85)(.15)}$

$= \sqrt{2.55} \approx 1.5969$.

n=6 p=.5 q=.5

2 Normal Distributions



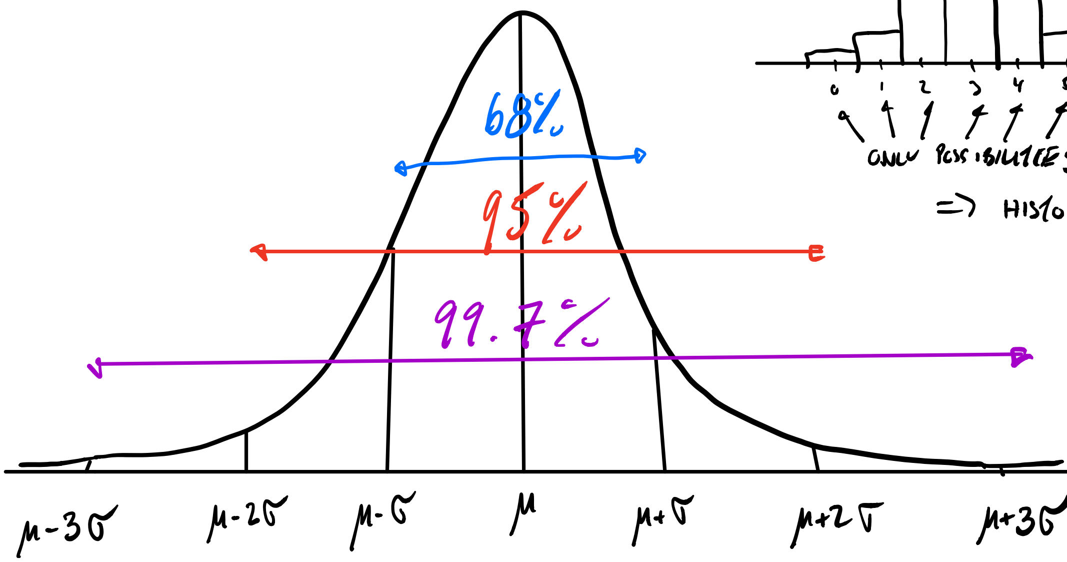
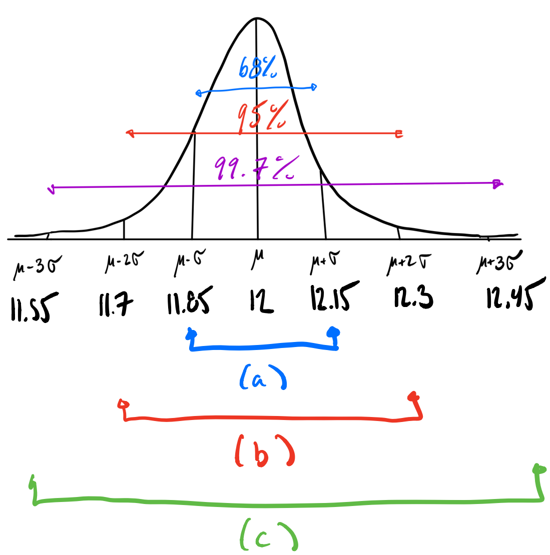


Figure 1: The 68-95-99.7 rule for normal distributions.

4. A machine in a bottling plant is set to dispense 12 oz of soda into cans. The machine is not perfect, and so every time the machine dispenses soda, the exact amount dispensed is a number x with a normal distriution. The mean and standard deviation for x are $\mu = 12$ oz and $\sigma = 0.15$ oz, respectively. Approximate the following probabilities using the 68-95-99.7 rule.
   (a) Give a range of values such that the amount of soda in 68% of all cans filled by this machine are in this range.
   (b) Give a range of values such that the amount of soda in 95% of all cans filled by this machine are in this range.
   (c) Give a range of values such that the amount of soda in 99.7% of all cans filled by this machine are in this range.
   (d) $P(11.85 \le x \le 12.15)$ .68
   (e) $P(11.70 \le x \le 12)$ .475
   (f) $P(x \le 11.70)$
   (g) $P(12.3 \le x \le 12.45)$
   (h) $P(x \le 12 \cup x \ge 12.45)$



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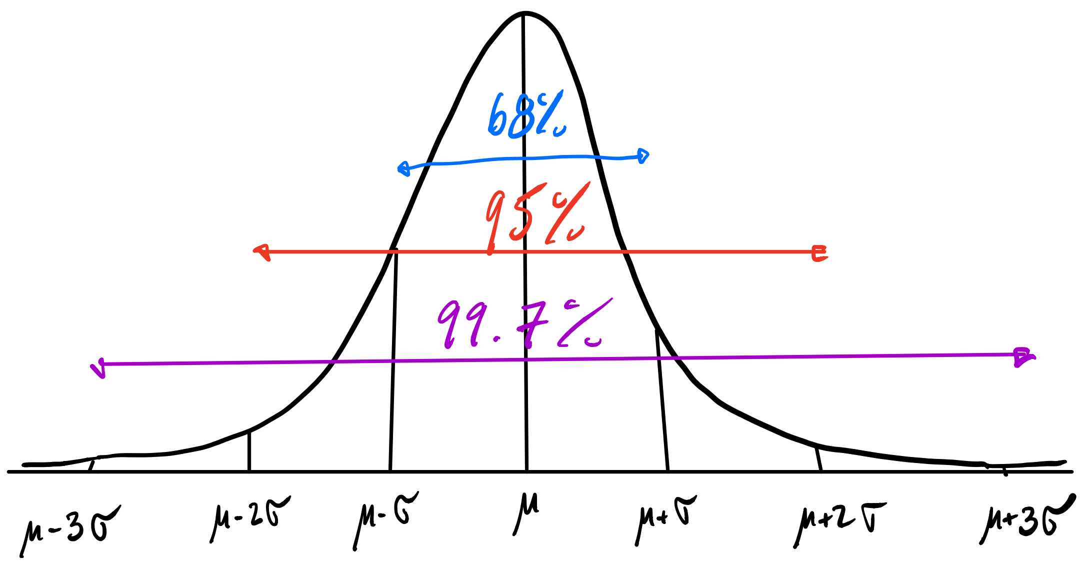

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