# Module 7: Conditional Probability

**SUMMARY** **Key Concepts**

***Conditional Probability***

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

**Note:** $P(A|B)$ is a probability based on the new sample space $B$, while $P(A \cap B)$ and $P(B)$ are probabilities based on the original sample space $S$.

***Product Rule***

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

***Independent Events***

- $A$ and $B$ are **independent** if

$$P(A \cap B) = P(A)P(B)$$

- If $A$ and $B$ are independent events with nonzero probabilities, then

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

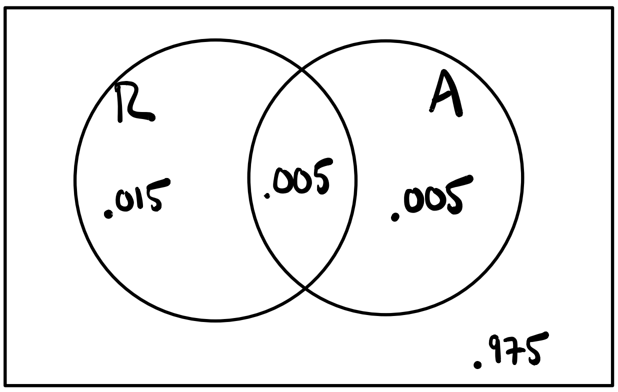
- If $A$ and $B$ are events with nonzero probabilities and either $P(A|B) = P(A)$ or $P(B|A) = P(B)$, then $A$ and $B$ are independent.
- If $E_1, E_2, \ldots, E_n$ are independent, then

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2) \cdot \cdots \cdot P(E_n)$$

Example. A car insurance company kept detailed records on all of its 1000 policy holders during 2019. The records show that most of their policy holders obeyed traffic laws and drove safe. In fact, during 2019, only 2% of policy holders received points on their license for a moving violation, 1% of policy holders got into a major accident, and only 0.5% of policy holders both received points on their license for a moving violation and got into a major accident.

1. What is the probability that a policy holder gets into a major accident?
2. What is the probability that a policy holder with points on their license gets into a major accident?
3. What is the probability that a policy holder who gets into a major accident has points on their license?

PROB



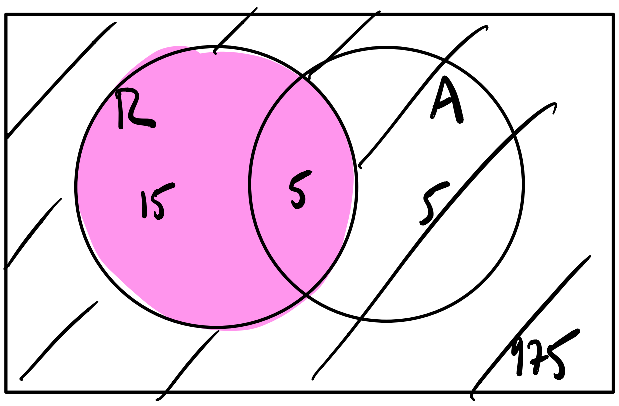

R = RECEIVE POINTS ON LICENSE

A = ACCIDENT

1. $P(A) = .01$

2. RESTRICTED SAMPLE SPACE TO ONLY POLICY HOLDER WITH POINTS ON LICENSE (20 PEOPLE)

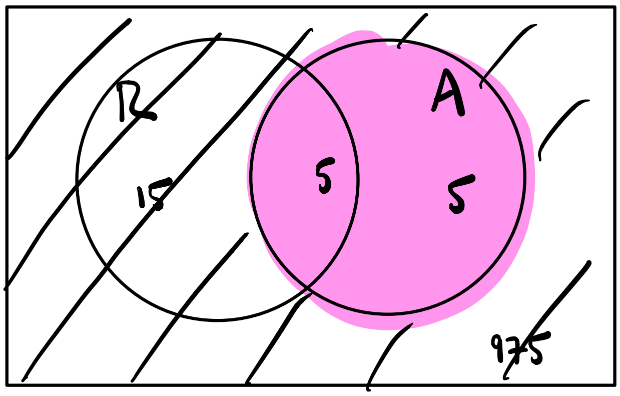
\# PEOPLE




$$P(A|R) = \frac{5}{20}$$

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{.005}{.02} = .25$$

3.




PROB THAT A PERSON WHO GETS IN ACCIDENT HAS RECEIVED POINTS ON LICENSE.

$$P(R \mid A) = \frac{5}{10} = .5$$

$$P(R \mid A) = \frac{P(R \cap A)}{P(A)} = \frac{.005}{.01} = .5$$

Example. You know that your neighbors have two children. Then one day you see one of the children playing outside, and this child is a boy. What is the probability that the other child is also a boy?

X SAMPLE SPACE $S = \{ \underbrace{2B,\ 2G,\ 1B1G}_{\text{NOT ALL EQUALLY LIKELY}} \}$

✓ SAMPLE SPACE $S = \{ \underbrace{BB,\ BG,\ GB,\ GG}_{\text{ALL EQUALLY LIKELY}} \}$

$$P(BB) = \frac{1}{4}$$

SAMPLE SPACE GETS UPDATED BY NEW INFO!

CONDITIONAL PROB.

$P(BB \mid GG')$

$= \frac{1}{3}$

$S = \{ \underbrace{BB, BG, GB}_{\text{ALL EQUALLY LIKELY}}, \cancel{GG} \}$

DEF OF CONDITIONAL PROBABILITY

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

EQUIVALENT EQUATIONS

$$P(B) \times P(A \mid B) = \frac{P(A \cap B)}{\cancel{P(B)}} \times \cancel{P(B)}$$

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)$$

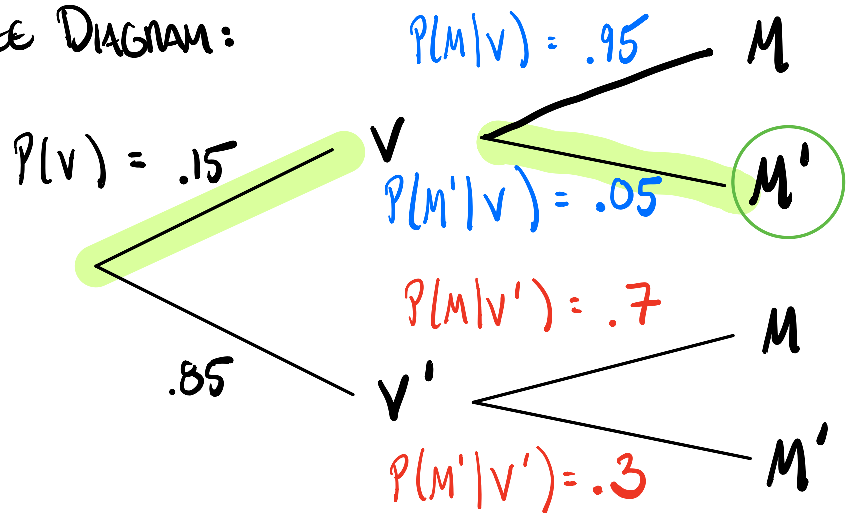
PRODUCT / MULTIPLICATION RULE

CONDITIONAL CLAUSE
=> CONDITIONAL PROBABILITIES

Example. Your company is about to begin a kickstarter campaign and wants to know the probability that you will be able to raise 1 million dollars in 8 weeks. You know that if the campaign video goes viral then the probability is 95%, and if the campaign does not go viral then there is a 70% chance. Suppose there is a 15% chance that the campaign video will go viral.

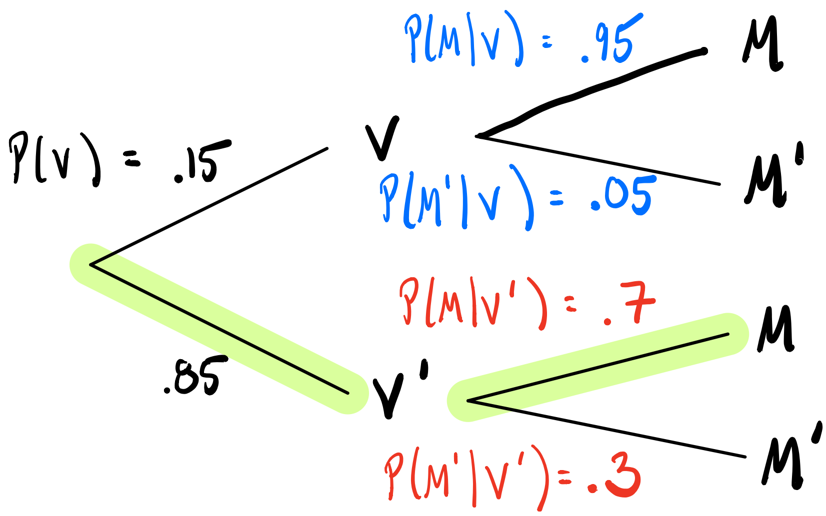
1. What is the probability that the campaign video goes viral but the campaign is not able to raise 1 million dollars?
2. What is the probability that the campaign does not go viral and the campaign still raises 1 million dollars?
3. What is the probability that the campaign raises 1 million dollars?

Tree Diagram:



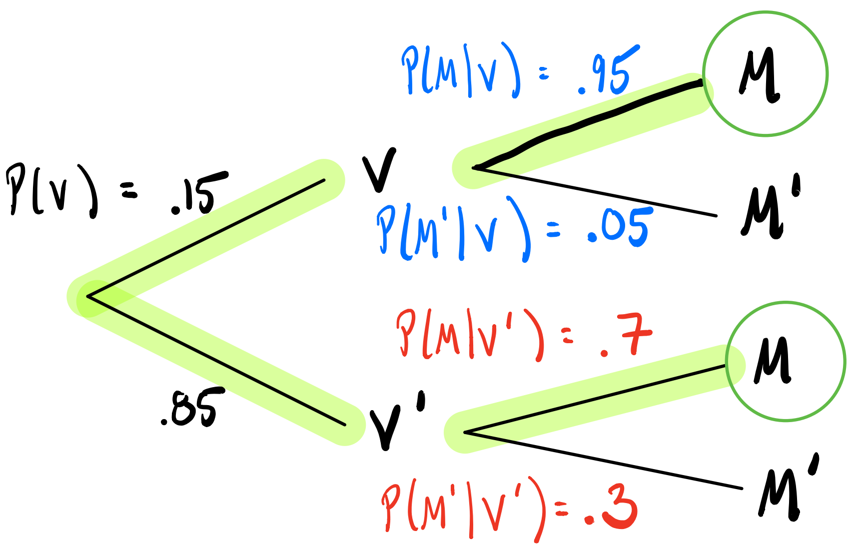

1.

$$P(V \cap M') = P(V)P(M'|V)$$
$$= (.15)(.05)$$
$$= .0075$$




2.

$$P(V' \cap M) = P(V')P(M|V')$$
$$= (.85)(.7)$$
$$= .595$$




3.

$$P(M) = P(V \cap M) + P(V' \cap M)$$
$$= P(V)P(M|V) + P(V')P(M|V')$$
$$= (.15)(.95) + (.85)(.7)$$
$$= .1425 + .595$$
$$= .7375$$

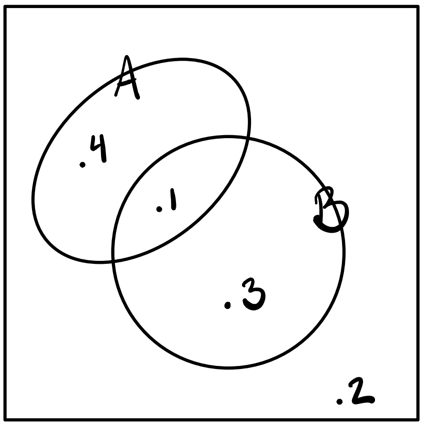
# Independent Events

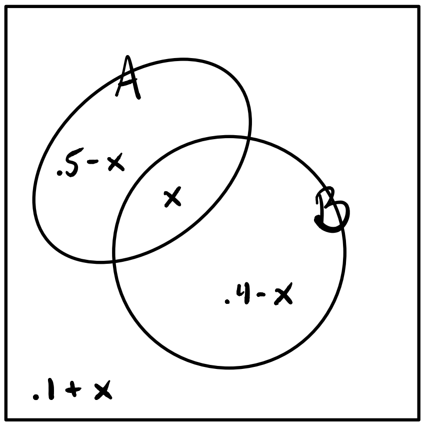
Given 2 events $A$, $B$.

$P(A) = .5$

$P(B) = .4$

Do we know $P(A \cap B)$? **No.**

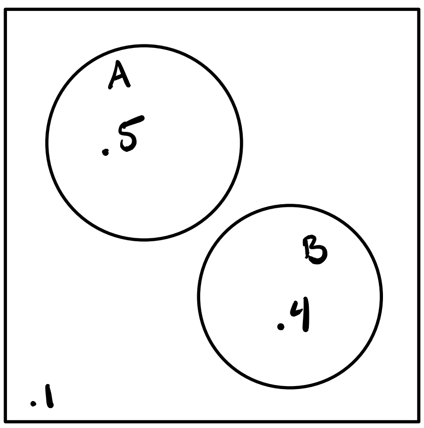


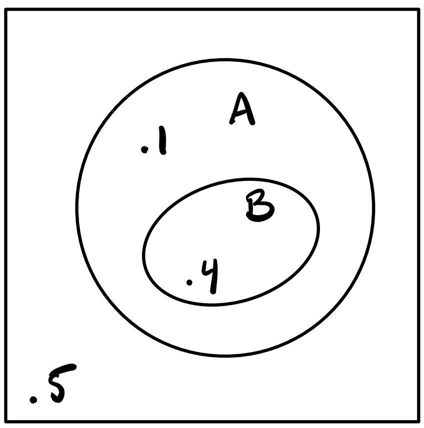




$0 \le x \le .4$

$P(A) = .5$

$P(A|B) = 0$




$P(A) = .5$

$P(A|B) = 1$

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

unknown

Def: 2 events $A, B \subseteq S$ are **independent** if & only if $P(A|B) = P(A)$ and
$P(B|A) = P(B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$ special case!

**74.** Ann and Barbara are playing a tennis match. The first player to win 2 sets wins the match. For any given set, the probability that Ann wins that set is $\frac{2}{3}$. Find the probability that

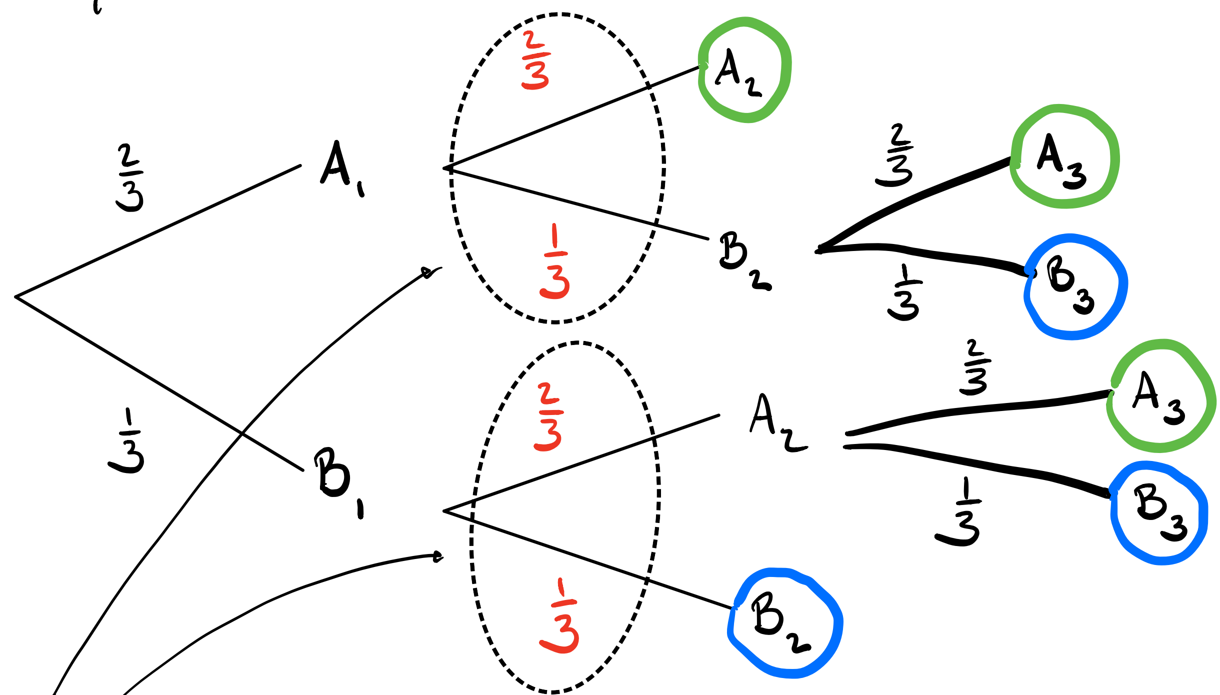
(A) Ann wins the match.

(B) 3 sets are played.

(C) The player who wins the first set goes on to win the match.

Let $A_i$ = Ann wins $i^{th}$ set

$B_i$ = Barbara wins $i^{th}$ set




Identical sets of probabilities

$$P(A_2|A_1) = P(A_2|B_1) = \frac{2}{3}$$

$$P(\text{ANN WINS GAME}) = P(A_1 A_2) + P(A_1 B_2 A_3) + P(B_1 A_2 A_3)$$

$$= \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)$$

$$= .7407$$

**THEOREM 1 Bayes' Formula**

Let $U_1, U_2, \ldots, U_n$ be $n$ mutually exclusive events whose union is the sample space $S$. Let $E$ be an arbitrary event in $S$ such that $P(E) \neq 0$. Then,

$$P(U_1|E) = \frac{P(U_1 \cap E)}{P(E)} = \frac{P(U_1 \cap E)}{P(U_1 \cap E) + P(U_2 \cap E) + \cdots + P(U_n \cap E)}$$

$$= \frac{P(E|U_1)P(U_1)}{P(E|U_1)P(U_1) + P(E|U_2)P(U_2) + \cdots + P(E|U_n)P(U_n)}$$

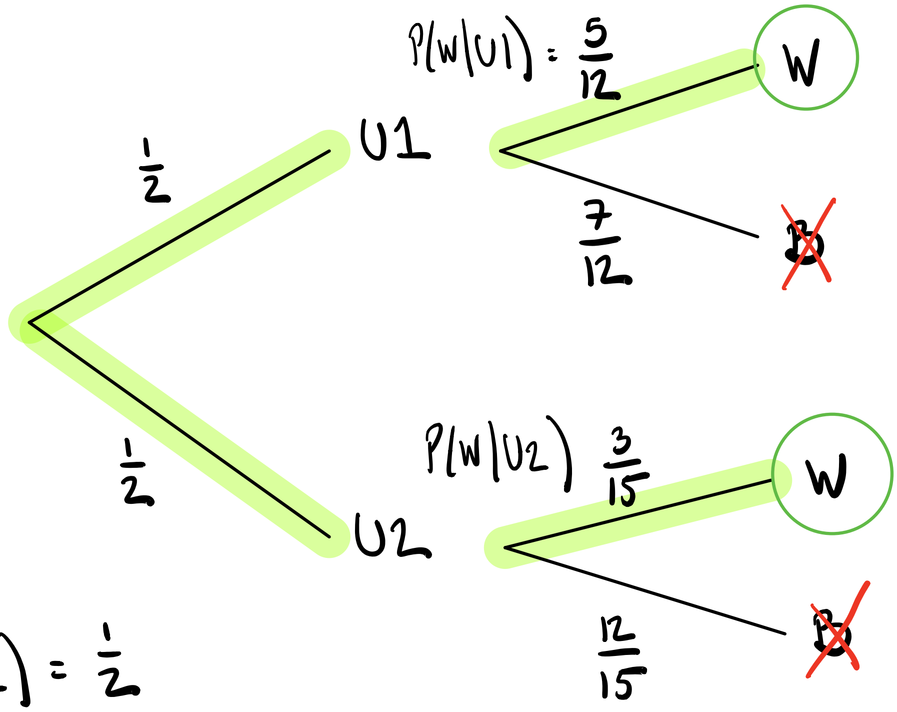
Similar results hold for $U_2, U_3, \ldots, U_n$.

DEF. OF CONDITIONAL PROB. + PRODUCT RULE

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \begin{cases} \dfrac{P(A)P(B|A)}{P(B)} \\[2ex] \dfrac{\cancel{P(B)}\,P(A|B)}{\cancel{P(B)}} \end{cases}$$

BAYES' RULE $\quad P(A|B) = \dfrac{P(A)P(B|A)}{P(B)}$

Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 whites and 12 black. A fair coin is flipped; if it is Heads, a ball is drawn from Urn 1, and if it is Tails, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was in fact taken from Urn 2? (i.e., that the coin flip was Tails)




$P(U2) = \frac{1}{2}$

FIND $P(U2|W)$

BAYES' RULE : $P(U2|W) = \dfrac{P(U2)P(W|U_2)}{P(W)}$

$$P(U2|W) = \frac{P(U2)P(W|U_2)}{P(U1 \cap W) + P(U2 \cap W)}$$

$$P(U2|W) = \frac{(\frac{1}{2})(\frac{3}{15})}{(\frac{1}{2})(\frac{5}{12}) + (\frac{1}{2})(\frac{3}{15})} = .3243$$

Example. One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time. (a). What is the probability that Joe (a random person) tests positive? (b). Joe just got the bad news that the test came back positive; what is the probability that Joe has the disease?

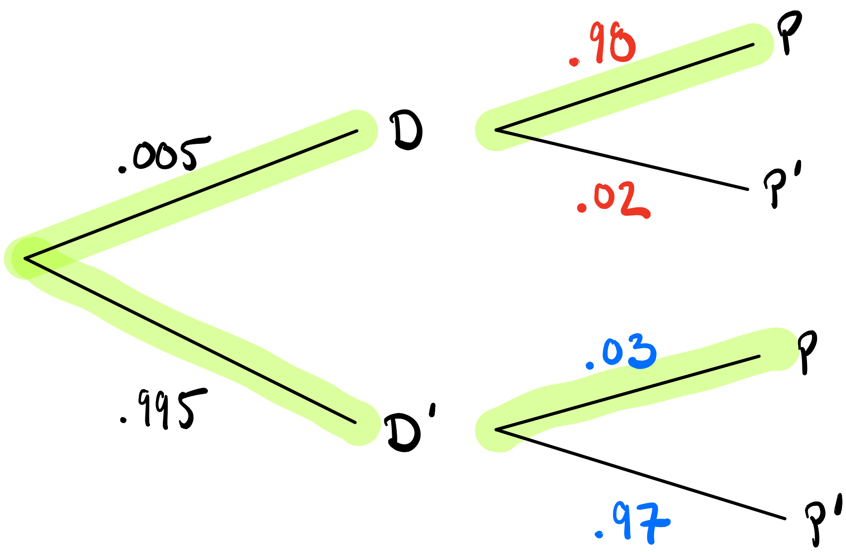
LET $D$ = PERSON HAS DISEASE $\quad D'$ = NO DISEASE

$P$ = PERSON TESTS POSITIVE $\quad P'$ = TESTS NEGATIVE

GIVEN: $P(D) = .005 \Rightarrow P(D') = .995$

$P(P \mid D') = .03 \Rightarrow P(P' \mid D') = .97$

$P(P' \mid D) = .02 \Rightarrow P(P \mid D) = .98$




(a) $P(P) = P(P \cap D) + P(P \cap D')$

$= P(D)P(P \mid D) + P(D')P(P \mid D')$

$= (.005)(.98) + (.995)(.03) = .03475$

(b) $P(D \mid P) = \dfrac{P(D)P(P \mid D)}{P(P)} = \dfrac{(.005)(.98)}{.03475} = .1410$

BAYES' RULE